Direct Detection of DM

1. **QCD and Nucleons**
The fundamental fields of QCD are quarks and gluons, but we know that these become strongly-coupled at energies $E \sim \Lambda_{QCD} \simeq 200$ MeV. At energies below this, the relevant degrees of freedom are nucleons and mesons. We can describe their low-energy interactions perturbatively by writing an effective field theory for them. The tricky part is that we don't know how to predict what these interactions should be from the underlying theory of quarks and gluons because of their very strong interactions. Even so, we can match many aspects between the two theories by making use of the underlying symmetry properties. The remaining gaps can be filled by lattice simulations of QCD.

   a) Let's apply these ideas to direct detection. Suppose the underlying DM-quark interactions are

$$-\mathscr{L} \supset \bar{\chi}\chi \left(\sum_{q=u,d,s} d_q\, \bar{q}q + \sum_{Q=c,b,t} d_Q\, \bar{Q}Q \right) . \tag{1}$$

   We want to convert this into an effective χ-nucleon interaction. For the light quark couplings, lattice QCD gives

$$\langle \tilde{n}(p_4)|\, m_q \bar{q}q\, |\tilde{n}(p_2)\rangle = m_{\tilde{n}} f_{T_q}^{(\tilde{n})}\, \bar{u}_4 u_2 \tag{2}$$

   for some constants $f_{T_q}^{(\tilde{n})}$, and $\bar{u}_4 = \bar{u}(p_4, s_4)$ and $u_2 = u(p_2, s_2)$ are polarization spinors. We also know that $\langle \tilde{n}(p_4)|\, \bar{\tilde{n}}\tilde{n}\, |\tilde{n}(p_2)\rangle = \bar{u}_4 u_2$ for nucleon fields. In the case of $d_Q = 0$ for all the heavy quarks, what χ-nucleon effective interaction would reproduce the effects of the underlying quark interactions (*i.e.* give the same matrix elements between nucleon states) given Eq. (2)?

   b) In general, there will also be a contribution to the effective χ-nucleon interaction from heavy quarks. The effect of integrating them out at one-loop order is to generate the effective interactions obtained by making the replacements

$$\bar{Q}Q \to -\frac{2\alpha_s}{24\pi\, m_Q}\, G^a_{\mu\nu}G^{a\,\mu\nu} \tag{3}$$

wherever $\bar{Q}Q$ appears in the Lagrangian. What is the resulting low-energy effective Lagrangian in terms of only the light quarks q and the gluon field $G^a_{\mu\nu}$?

c) Given that we also have an induced $\bar{\chi}\chi G^a_{\mu\nu}G^{a\,\mu\nu}$ interaction, we need to relate it as well to a nucleon interaction. Let us define

$$\langle\tilde{n}|\, G^a_{\mu\nu}G^{a\,\mu\nu}\,|\tilde{n}\rangle = -\frac{8\pi}{9\alpha_s}\, f^{(\tilde{n})}_{T_G} m_{\tilde{n}}\ . \tag{4}$$

It turns out that we can relate $f^{(\tilde{n})}_{T_G}$ to the constants for light quarks. To do this, we should look at the divergence of the *dilatation current*, which is just the trace of the improved energy-momentum tensor, $\Theta^\mu{}_\mu$ (see Ch.19.5 of Peskin&Schroeder). The Lagrangian of QCD is invariant under scale transformations ($x \to \lambda x$, $q \to \lambda^{-3/2}q$, $A^a_\mu \to \lambda^{-1}A^a_\mu$) when the quark masses vanish. Quantum effects also break the invariance in the form of the running of strong coupling. As a result, the divergence of the dilatation current is equal to these breaking effects:

$$\Theta^\mu{}_\mu = \sum_{q=u,d,s} m_q\bar{q}q - \frac{9\alpha_s}{8\pi}G^a_{\mu\nu}G^{a\,\mu\nu}\ . \tag{5}$$

The first term comes from the explicit breaking by quark masses while the second term comes from the quantum breaking effect, with the coefficient of the second piece being proportional to the beta function of QCD with three light flavours. We can also write a dilatation current for the low-energy theory with nucleon fields:

$$\Theta^\mu{}_\mu = m_p\bar{p}p + m_n\bar{n}n + \dots\ . \tag{6}$$

Since both descriptions of QCD describe the same underlying theory, the dilatation current operators in the two cases must be equivalent in the sense that they have the same matrix elements. Use this fact to relate $f^{(\tilde{n})}_{T_G}$ to the $f^{(\tilde{n})}_{T_q}$ constants.

d) Put all these pieces together to find the effective χ couplings to protons and neutrons.

2. **Spin-Dependent Cross Sections**
(Borrowed from notes by P. Salati: http://inspirehep.net/record/776274)

a) The matrix element for χ-nucleus scattering from the spin-dependent AA DM-quark interaction is

$$\mathcal{M} = 2\sqrt{2}G_F\Lambda_N\,\langle\chi(p_3,s_3)|\,\bar{\chi}\gamma^\mu\gamma^5\chi\,|\chi(p_1,s_1)\rangle\,\langle N(p_4;J,m_f)|\,S_\mu\,|N(p_2;J,m_i)\rangle\ . \tag{7}$$

Sum this over final states and average it over initial states, treating the DM particle χ as a fundamental fermion, the nucleus as a non-relativistic system with total spin J (and magnetic states $m = -J, -J+1, \dots, J$), and assuming that the operator S^μ is Hermitian, to show that:

$$\text{“}|\mathcal{M}|^2\text{”} = 8\kappa^2G_F^2\Lambda_N^2\,\frac{1}{2(2J+1)}\,\chi^{\mu\nu}N_{\mu\nu}\ , \tag{8}$$

where

$$\chi^{\mu\nu} = tr\left[(\not{p}_3 + m_\chi)\gamma^\mu\gamma^5(\not{p}_1 + m_\chi)\gamma^\nu\gamma^5\right] \tag{9}$$

and

$$N_{\mu\nu} = \sum_{m_i,m_f} \langle J, m_f| S_\mu |J, m_i\rangle \langle J, m_i| S_\nu |J, m_f\rangle \ . \tag{10}$$

b) Work out the trace for χ in the usual way, and simplify it in the lab frame in the extreme non-relativistic limit $v \to 0$ (so that $v \to 0$ and $p_1 = p_3 = (m_\chi, \vec{0})$. You can also drop any terms that will give zero when you contract with the symmetric tensor $N_{\mu\nu}$.

c) Simplify and evaluate $N_{\mu\nu}X^{\mu\nu}$. For this, use the fact that $S^\mu \to (0, \vec{J})$ in the non-relativistic limit, where $\vec{J}$ is the familiar spin operator, and apply the simple form of $\chi^{\mu\nu}$ found in b). Note also that for the nuclear matrix elements, we are implicitly assuming a relativistic normalization of $\langle J, m_1|J, m_2\rangle = 2E\delta_{m_1,m_2}$, where E is the energy of the state.

d) Put everything together to show that "$|\mathcal{M}|^2$" $\propto G_F^2\Lambda_N^2\, J(J+1)$.