Effective Lagrangians

1 Effective Field Theories

a) Consider the interactions

$$-\mathscr{L} \supset g_\chi \bar{\chi}\chi\,\phi + g_f \bar{f} f \phi \tag{1}$$

where χ is a scalar field with mass $m_\phi \gg m_\chi \sim m_f$. Compute the summed and squared matrix element for $\chi f \to \chi f$ from these interactions in the lab frame (where χ is incident upon f).

b) In the previous tutorial you computed the summed and squared matrix element (in the lab frame) for the interaction

$$-\mathscr{L} \supset d_q (\bar{\chi}\chi)(\bar{q}q) \tag{2}$$

Compare this result to part a) in the two limits $|\vec{p}| \gg m_\phi$ and $|\vec{p}| \ll m_\phi$, where $\vec{p}$ is the 3-momentum of the incoming χ.

c) In part b) you should find that the matrix elements have the same energy dependence for $|\vec{p}| \ll m_\phi$. What is the value of d_q that would allow the interaction of Eq. (2) to reproduce (at leading order in $|\vec{p}|/m_\phi$) the result predicted by Eq. (1).

This example illustrates an effective field theory. At energies well below the mass of ϕ we can remove this particle from the theory and replace it with effective interactions involving only the light ($m \lesssim E$) fields in the theory. This is what Eq. (2) does for the full theory described by Eq. (1) to leading order in an expansion in powers of $|\vec{p}|/m_\phi$. The process of removing heavy particles and replacing them with new interactions is called integrating out. Another well-known example is Fermi's four-fermion theory of the weak interactions, which is obtained by integrating out the W and Z vector bosons.

2 QCD and EFTs

The fundamental fields of QCD are quarks and gluons, but we know that these become strongly-coupled at energies $E \sim \Lambda_{QCD} \simeq 1$ GeV. At energies below this, the relevant degrees of freedom are nucleons and mesons. We can describe their low-energy interactions perturbatively by writing an effective field theory for them. The tricky part is that we don't know how to predict what these interactions should be from the underlying theory of quarks and gluons because of their very strong interactions. Even so, we can match many aspects between the two theories by making use of the underlying symmetry properties. The remaining gaps can be filled by lattice simulations of QCD.

a) Let's apply these ideas to direct detection. Suppose the underlying DM-quark interactions are

$$-\mathscr{L} \supset \bar{\chi}\chi \left(\sum_{q=u,d,s} d_q\, \bar{q}q + \sum_{Q=c,b,t} d_Q\, \bar{Q}Q \right) . \tag{3}$$

We want to convert this into an effective χ-nucleon interaction. For the light quark couplings, lattice QCD gives

$$\langle \tilde{n}|\, m_q \bar{q}q \,|\tilde{n}\rangle = m_{\tilde{n}} f_{T_q}^{(\tilde{n})} \tag{4}$$

for some constants $f_{T_q}^{(\tilde{n})}$. We also know that $\langle \tilde{n}|\, \bar{\tilde{n}}\tilde{n} \,|\tilde{n}\rangle = 1$. In the case of $d_Q = 0$ for all the heavy quarks, what χ-nucleon effective interaction would reproduce the effects of the underlying quark interactions (*i.e.* give the same matrix elements between nucleon states) given Eq. (4)?

b) In general, there will also be a contribution to the effective χ-nucleon interaction from heavy quarks. The effect of integrating them out at one-loop order is to generate the effective interactions obtained by making the replacements

$$\bar{Q}Q \to -\frac{2\alpha_s}{24\pi\, m_Q}\, G^a_{\mu\nu} G^{a\,\mu\nu} \tag{5}$$

wherever $\bar{Q}Q$ appears in the Lagrangian. What is the resulting low-energy effective Lagrangian in terms of only the light quarks q and the gluon field $G^a_{\mu\nu}$.

c) Given that we also have an induced $\bar{\chi}\chi G^a_{\mu\nu} G^{a\,\mu\nu}$ interaction, we need to relate it as well to a nucleon interaction. Let us define

$$\langle \tilde{n}|\, G^a_{\mu\nu} G^{a\,\mu\nu} \,|\tilde{n}\rangle = -\frac{8}{9\pi\alpha_s}\, f_{T_G}^{(\tilde{n})} . \tag{6}$$

It turns out that we can relate $f_{T_G}^{(\tilde{n})}$ to the constants for light quarks. To do this, we should look at the divergence of the *dilatation current*, which is just the trace of the improved energy-momentum tensor, $\Theta^{\mu}{}_{\mu}$ (see Ch.19.5 of Peskin&Schroeder). The Lagrangian of QCD is invariant under scale transformations ($x \to \lambda x$, $q \to \lambda^{-3/2} q$, $A^a_\mu \to \lambda^{-1} A^a_\mu$) when the quark masses vanish. Quantum effects also break the invariance in the form of the running of strong coupling. As a result, the divergence of the dilatation current is equal to these breaking effects:

$$\Theta^{\mu}{}_{\mu} = \sum_{q=u,d,s} m_q \bar{q}q - \frac{9\alpha_s}{8\pi} G^a_{\mu\nu} G^{a\,\mu\nu} . \tag{7}$$

The first term comes from the explicit breaking by quark masses while the second term comes from the quantum breaking effect, with the coefficient of the second piece being proportional to the beta function of QCD with three light flavours. We can also write a dilatation current for the low-energy theory with nucleon fields:

$$\Theta^{\mu}{}_{\mu} = m_p \bar{p}p + m_n \bar{n}n + \dots . \tag{8}$$

Since both descriptions of QCD describe the same underlying theory, the dilatation current operators in the two cases must be equivalent in the sense that they have the same matrix elements. Use this fact to relate $f_{T_G}^{(\tilde{n})}$ to the $f_{T_q}^{(\tilde{n})}$ constants.

d) Put all these pieces together to find the effective χ couplings to protons and neutrons.