Tutorial # 3

1. **R-Symmetries**
   There is one class of exceptions to the claim in `notes-02` that the generators of gauge or global symmetries must commute with the supersymmetry generators. These exceptions are called an R-symmetries. For $\mathcal{N}=1$ supersymmetry in $d=4$ dimensions, the maximal possibility is a global $U(1)_R$, and its generator R commutes with P^μ and $J^{\mu\nu}$ and satisfies
$$[R,Q] = -Q\ , \qquad [R,\overline{Q}] = +\overline{Q}\ .$$
   a) For chiral multiplets $\Phi = (\phi,\psi,F)$, the R-charge r_Φ of the multiplet corresponds to the charge of ϕ under $U(1)_R$ transformations, $\phi(x)\to e^{ir_\Phi\alpha}\phi(x)$. Let us denote the R-charge in this case by $[\phi]_R = r_\Phi$. The commutator of R with Q then implies that $[\psi]_R = (r_\Phi - 1)$, $[F]_R = (r_\Phi-2)$, and $[W]_R = 2$. For the superpotential $W=\lambda\Phi^3$, show that the theory has an R-symmetry provided $[\Phi]_R = 2/3$.
   Hint: to do so, show that $[W]_R = 2$ and that the terms in the Lagrangian derived from it, $F\ (\partial W/\partial\Phi)|_\phi$ and $(\partial^2W/\partial\Phi^2)|_\phi\,\psi\psi$, as well as the kinetic terms, are invariant this transformation.

   b) Generalize this result to a general superpotential W that is a polynomial function of the chiral superfields $\{\Phi_i\}$ with R-charges r_i such that all the terms in W have net R-charge of two. For this, note that the general set of Lagrangian terms derived from such a superpotential are
$$-\mathscr{L}\ \supset\ -\frac{1}{2}\sum_{i,j}\frac{\partial^2 W}{\partial\Phi_i\partial\Phi_j}\bigg|_\phi \psi_i\psi_j + \sum_i F_i\,\frac{\partial W}{\partial\Phi_i}\bigg|_\phi + (h.c.)\ ,$$
   where $|_\phi$ means to evaluate the expressions on the scalar components, $\Phi_i\to\phi_i$.
   Hint: show that for any given term in W, $[\partial W/\partial\Phi_i|_\phi]_R = (2-r_i)$ and generalize.

   c) For vector superfields $V = (\lambda, A^\mu, D)$, we have $[\lambda]_R = 1$, $[A^\mu] = [D] = 0$. Show that the Lagrangian interaction terms between vector and chiral multiplets, $\sqrt{2}g(\phi^\dagger t^a\psi)\lambda^a$ and $(\phi^\dagger t^a\phi)D^a$ are R-invariant as well.

   d) Find a set of R-charges under which the exactly supersymmetric part of the MSSM is invariant. Show as well that some of the terms in the soft breaking Lagrangian explicitly break this symmetry.

   e) Prove that R-parity in the MSSM can be identified with a discrete subgroup of a $U(1)_R$ symmetry of the supersymmetric part of the MSSM. Show as well that all the soft terms are invariant under this subgroup, even though they are not all invariant under the full $U(1)_R$.

2. **SUSY QED and Supersymmetry Breaking**
   There is an additional term that can be added to Abelian gauge theories that is consistent with supersymmetry. It is called a Fayet-Iliopoulos (FI) term ξ, and it modifies the scalar potential by $D\to D-\xi$, where ξ is a real constant.

a) Find the scalar potential of SUSY QED in the presence of a FI term. What is the space of vacua for $m = 0$?

b) Find the minimum of the potential for $m \neq 0$ and $m^2 > e^2\xi$ (with ξ assumed to be positive). Show that the vacuum energy is positive and compute the masses of physical states in the theory. What does this imply about supersymmetry?

c) Repeat part b) with $m^2 < e^2\xi$.
Hint: choose a gauge such that $\widetilde{E}$ *is real and expand* $E = \eta + h/\sqrt{2}$*, where* η *is the VEV and* h *is a real scalar. Also, remember the* $\sqrt{2}e\lambda E\widetilde{E}^*$ *term.*

d) You should have found that supersymmetry was sponteneously broken in parts b) and c). The massless fermion in both cases is the *goldstino*. It is analagous to a Nambu-Goldstone boson, but it is fermionic because the symmetry that has been spontaneously broken is fermionic. Despite this breaking, there is still a mass sum rule for the states. It is

$$Str(\mathcal{M}^2) \ := \ \sum_b g_b m_b^2 - \sum_f g_f m_f^2 \ = \ 0 \ ,$$

where the sums run over all the bosons (b) and fermions (f) in the theory, g_i is the number of real degrees of freedom of state i. Show that this sum rule is satisfied by the SUSY QED theory for ($\xi = 0, m \neq 0$) where SUSY is preserved, as well as in cases b) and c) above where SUSY is spontaneously broken.
Hint: a Weyl fermion (plus antiparticle) have $g_f = 2$*, a Dirac fermion has* $g_f = 4$*, a massless gauge boson has* $g_b = 2$*, a massive gauge boson has* $g_b = 3$*, and a complex scalar has* $g_b = 2$*.*