PSI BSM Homework #2

1. **Toy Superspace**
   Supersymmetry ($\mathcal{N}=1$, $d=4$) can be formulated in a very nice way on a *superspace* consisting of coordinates $(x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$, where θ and $\bar{\theta}$ are Grassmann-valued and transform as Weyl spinors under Lorentz. Instead of working with this full superspace, let's consider a simplified version in one dimension that retains a few of the key features of the full story.

   a) The translation part of the Poincaré group is realized as a *coordinate representation* on fields. Under translations by a^μ, fields transform as $\phi(x) \to \phi(x-a)$. Show that $P_\mu = i\partial_\mu$ ($\mu = 0,1,2,3$) generate the translation subgroup of Poincaré in the sense that

   $$\begin{aligned} 0 &= [P_\mu,\, P_\nu] \\ f(x) &\to e^{ia^\mu P_\mu} f(x) = f(x-a) \ . \end{aligned}$$

   b) Let's now specialize to one spacetime dimension ($d=1$) and define a toy superspace with coordinates $(x, \theta, \bar{\theta})$, where θ and $\bar{\theta}$ are independent Grassmann numbers (with no Lorentz indices since there are no boosts or rotations in $d=1$). We define the superspace operators P, Q, and $\bar{Q}$ by

   $$P = i\partial_x \ , \qquad Q = i\partial_\theta - \bar{\theta}\partial_x \ , \qquad \bar{Q} = -i\partial_{\bar{\theta}} + \theta\partial_x \ ,$$

   where $\partial_x = \partial/\partial x$, $\partial_\theta = \partial/\partial\theta$, and $\partial_{\bar{\theta}} = \partial/\partial\bar{\theta}$. Show that as operators acting on an arbitrary function $F(x,\theta,\bar{\theta})$, these satisfy the relations

   $$\begin{aligned} \{Q, \bar{Q}\} &= 2P \\ \{Q, Q\} &= 0 = \{\bar{Q}, \bar{Q}\} \\ [P, Q] &= 0 = [P, \bar{Q}] = [P, P] \ . \end{aligned}$$

   This gives a closed supersymmetry algebra.
   Hint: recall that for independent Grassmann variables η and ξ,

   $$\begin{aligned} \eta^2 &= 0 = \xi^2 \ , & \eta\xi &= -\xi\eta \ , \\ \partial_\eta \eta &= 1 = \partial_\xi \xi \ , & \partial_\eta \xi &= 0 = \partial_\xi \eta \\ \partial_\eta(\eta\xi) &= \xi = -\partial_\eta(\xi\eta) \ , & \partial_\eta\partial_\xi &= -\partial_\xi\partial_\eta \ . \end{aligned}$$

   c) A general function on superspace can be expanded in its fermionic coordinates, and must take the form

   $$\Phi(x,\theta,\theta) = \phi(x) + \theta\,\psi(x) + \bar{\theta}\,\bar{\chi}(x) + \theta\bar{\theta}\,F(x) \ ,$$

   with scalar functions ϕ and F and Grassmann-valued functions ψ and $\bar{\chi}$. A general supersymmetry transformation takes the form

   $$\Phi(x,\theta,\bar{\theta}) \to e^{-i(\xi Q - \bar{\xi}\bar{Q})}\Phi(x,\theta,\bar{\theta}) = \left[1 - i(\xi Q - \bar{\xi}\bar{Q}) + \ldots\right]\Phi(x,\theta,\bar{\theta}) \ .$$

Show that to linear order in the transformation parameters ξ and $\bar{\xi}$, the changes in the component fields are

$$\begin{aligned}\delta\phi &= \xi\psi + \bar{\xi}\bar{\chi}\ , \qquad \delta F = -i\xi\partial_x\psi + i\bar{\xi}\partial_x\bar{\chi} \\ \delta\psi &= (-i\partial_x\phi + F)\,\bar{\xi}\ , \qquad \delta\bar{\chi} = (-i\partial_x\phi - F)\,\xi\ .\end{aligned}$$

With some extra work (optional), it is possible to show that this transformation corresponds to a translation in superspace by (something like, up to signs) $\Phi(x,\theta,\bar{\theta}) \ \to\ \Phi(x - (\bar{\xi}\theta + \xi\bar{\theta}), \theta - \xi, \bar{\theta} - \bar{\xi})$.

d) Define the superspace operator $\bar{D}$ by

$$\bar{D} = \partial_{\bar{\theta}} - i\theta\partial_x\ .$$

Show that $\{\bar{D}, Q\} = 0 = \{\bar{D}, \bar{Q}\}$.

e) A *chiral* superfield is a general function on superspace subject to the constraint $\bar{D}\Phi = 0$. Show that this is satisfied provided $\bar{\chi} = 0$ and $F + i\partial_x\phi = 0$. Show also that these conditions are maintained by SUSY transformations.

f) We are now in a position to build actions that are invariant under supersymmetry.

i) Show that up to surface terms, the following combination is unchanged under supersymmetric transformations:

$$\int dx \int d\theta \int d\bar{\theta}\ \Phi(x,\theta,\bar{\theta})\ ,$$

where Φ is any superfield whatsoever.

ii) Show that a product of any two chiral superfields is also chiral. Generalize this to show that any holomorphic polynomial of chiral superfields is chiral.

iii) If Φ is a chiral superfield, show that the following is invariant under supersymmetry:

$$\int dx \int d\theta\ \Phi^n\ ,$$

where n is some positive integer. Also, work out its form in components.

iv) Work out the following supersymmetric invariant in terms of the component fields of the chiral superfield Φ:

$$\int dx \int d\theta \int d\bar{\theta}\ \Phi^\dagger\Phi\ .$$

This toy model can be generalized to a superspace formulation of $\mathcal{N} = 1$ in $d = 4$. The coordinates are now x^μ, θ_α, and $\bar{\theta}^{\dot{\alpha}}$, and the invariant action is

$$S = \int d^4x \int d^2\theta \int d^2\bar{\theta}\ \Phi^\dagger\Phi + \int d^4x \int d^2\theta\ W(\Phi) + (h.c.)\ .$$

Expanding in components of Φ, the first term gives kinetic terms and the second yields the interactions derived from the superpotential.

2. **Linear and Non-Linear Sigma Models**
Consider a theory of four real scalars written in terms of a complex (2×2) matrix-valued field S of the form

$$S = \sigma(x)\mathbb{I}/2 + i\alpha^a(x)t^a,$$

where $\mathbb{I}$ is the (2×2) unit matrix, $t^a = \sigma^a/2$ $(a \geq 1)$, and σ, α^a are real scalars. Take the Lagrangian to be

$$\mathscr{L} = tr(\partial_\mu S^\dagger \partial^\mu S) - \frac{\lambda}{2}\left[tr(S^\dagger S) - v^2/2\right]^2.$$

This renormalizable theory is an example of a *linear sigma model*.

a) Show that this theory has a global $SU(2)_L \times SU(2)_R$ symmetry under which the matrix field transforms as $S \to S' = LSR^\dagger$.

b) Show that one of the global minima of the potential is

$$\langle S \rangle = (v/2)\,\mathbb{I}\ .$$

What fraction of the original symmetry does this spontaneously break? How many NGBs will there be?

c) Let us choose to expand around this "identity" vacuum state. One way to do so is write

$$S(x) = [v + h(x)]/2 + i\alpha^a(x)t^a\ .$$

Check that the kinetic terms are canonical and find the masses of these fields.

d) Another way to expand around the vacuum is to write

$$S(x) = \frac{1}{2}[v + H(x)]\exp[2i\pi^a(x)t^a/f]\ ,$$

for some f with mass dimension one. The exponential term can be thought of as a general transformation on the identity vacuum. For this choice of field variables:

i) Work out the kinetic terms for these new field variables, and fix f such that they become canonical. What are the masses of the H and π^a excitations?

ii) Suppose we are interested in physics at energies well below the mass of the H field. Work out the leading terms in the low-energy EFT you get by integrating out H. The result is sometimes called a *non-linear sigma model*.

e) Couple S to a doublet of Dirac fermions,

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix},$$

with $\psi_L \to L\psi_L$ and $\psi_R \to R\psi_R$ under $SU(2)_L \times SU(2)_R$. These symmetries forbid bare fermion masses, but we can still generate masses consistent with the symmetries by coupling to S:

$$-\mathscr{L} \supset g\,\bar{\psi}_L S\psi_R + h.c.$$

Work out the couplings in terms of the exponential representation of S discussed above. What are the resulting fermion masses? As the notation here might suggest, this theory was invented as an early model for the interactions of pions and nucleons.

Totally Optional Extra Problem: Operator Scaling

We are often interested in using theories defined at one scale to make predictions at a very different energy scale. In this case it is often very helpful to extrapolate from one scale to another using the renormalization group (RG). Let us assume that we start at scale μ_0 and extrapolate to scale μ in a theory whose interactions are dominated by gauge interactions of strength $\alpha(\mu) = g^2(\mu)/4\pi$.

a) Suppose the evolution of the gauge coupling is given by

$$\frac{d\alpha}{dt} = -b\,\alpha^2(t) \ ,$$

where $t = \ln(\mu/\mu_0)$ and b is a non-zero constant. Solve this equation for $\alpha(\mu)$ in terms of $\alpha(\mu_0)$.

b) The gauge coupling is the coefficient of just one operator in the effective Lagrangian; we can also apply RG to other operators. Suppose we have

$$-\mathscr{L} \supset \zeta\, M^{4-n} \mathcal{O}_\zeta^{(n)} \ ,$$

where $\mathcal{O}_\zeta^{(n)}$ is an operator with a (classical) mass dimension of n and M is a fixed dimension-one constant. Just like the gauge coupling, the coefficient ζ can be RG-evolved. Suppose its evolution equation is

$$\frac{1}{\zeta}\frac{d\zeta}{dt} = \gamma_\zeta(\alpha(t)) \ .$$

Show that a solution to this equation for $\zeta(\mu)$ is

$$\zeta(\mu) = \zeta(\mu_0)\ \exp\left[\int_{t_0}^{t} dt\,\gamma_\zeta(\alpha(t))\right]$$

c) For processes with typical momentum p, the relative contribution of the operator $\mathcal{O}_\zeta^{(n)}$ to the dynamics is

$$\zeta(\mu = p)\ \left(\frac{p}{M}\right)^{n-4}$$

For this reason, we say that n is the classical scaling dimension of the operator $\mathcal{O}_\zeta^{(n)}$. However, since quantum effects lead to the RG evolution of ζ, the full quantum evolution is a bit different.

i) Assume that

$$\gamma_\zeta(t) = -a\alpha(t) \ .$$

Use this and your solution for $\alpha(\mu)$ to solve for $\zeta(\mu)$ in terms of $\zeta(\mu_0)$ and $\alpha(\mu_0)$. What does this imply for the net momentum scaling of the operator?

ii) Suppose instead that the theory is approximately *conformal*, in the sense that $\alpha(\mu) = \alpha_*$ is constant. Show that this implies that $\gamma_\zeta = \gamma_*$ is constant as well, and solve for $\zeta(\mu)$ in this case. What does this imply for the momentum scaling of the operator?

d) Suppose we include higher-order corrections to the RG equation for α, and that these take the form

$$\frac{d\alpha}{dt} = -b_1\alpha^2 + b_2\alpha^3 \ ,$$

with b_1, $b_2 > 0$. For what special value of $\alpha = \alpha_*$ does $d\alpha/dt$ vanish? What happens if we start at $\mu_0 > \mu$ with $\alpha(\mu_0) > \alpha_*$ and evolve down to lower energies? Similarly, what happens if we start at $\mu_0 > \mu$ with $\alpha(\mu_0) < \alpha_*$ and evolve down? The special value α_* is sometimes called a non-trivial IR fixed point.