# PSI BSM Homework #1

1. **Generating Scalar Interactions**
   Consider the theory

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 + \overline{\psi} i\gamma^\mu \partial_\mu \psi - M\overline{\psi}\psi - y\phi\overline{\psi}\psi \ .$$

   a) Compute the divergent part of $\Delta\Gamma^{(\phi^3)}$ at one-loop using a cutoff regular in Euclidean space. Don't worry about factors of two or $\pi$, but make sure you get the full dependence on the couplings of the theory.

   b) Compute the divergent part of $\Delta\Gamma^{(\phi^4)}$ at one-loop using a cutoff regular in Euclidean space. Don't worry about factors of two or $\pi$, but make sure you get the full dependence on the couplings of the theory.

   c) Show that these results are consistent with the "pretend symmetry" of the theory discussed in class.

2. **The O'Raifeartaigh Model**
   Consider a supersymmetric theory of three chiral superfields $\Phi_1$, $\Phi_2$, and $\Phi_3$ with the superpotential

$$W = -t\Phi_1 + m\Phi_2\Phi_3 + \frac{\lambda}{2}\Phi_1\Phi_3^2 \ .$$

   Take the components of these to be $\Phi_i = (\phi_i, \psi_i, F_i)$, with $i = 1, 2, 3$, and assume that all couplings are real and positive and that $m^2 > \lambda\, t$.

   a) Work out the scalar potential and the fermion mass and Yukawa couplings. Show the result both before and after integrating out the $F_i$ terms.
   *Hint: the generalization to multiple superfields is*

$$\mathscr{L}_{int} \;=\; -\frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 W}{\partial\Phi_i\partial\Phi_j}\right|_\phi \psi_i\psi_j + \sum_i F_i \left.\frac{\partial W}{\partial\Phi_i}\right|_\phi + (h.c.)$$

   b) Show that it is not possible to have all $F_i = 0$ simultaneously and that the vacuum energy must be positive. This implies that supersymmetry is spontaneously broken.

   c) Demonstrate that turning on non-zero VEVs for $\phi_1$ and $\phi_2$ can only increase the potential energy. Find the minimum of the scalar potential for $\phi_1 = \phi_2 = 0$.
   *Hint: before minimizing, show that the minimum will occur with the VEV of $\phi_3$ real. Because of this, it sufficient to treat $\phi_3$ as a real variable when minimizing.*

   d) Expand around this minimum and find the masses of all the physical states.
   *Hint: expand $\phi_3$ in real and imaginary components and find the mass for each.*

e) Show that the *supertrace* vanishes in this theory even though supersymmetry is spontaneously broken. It is defined by

$$Str(\mathcal{M}^2) \;:=\; \sum_b g_b m_b^2 - \sum_f g_f m_f^2 \;=\; 0 \;,$$

where the sums run over all the bosons ($b$) and fermions ($f$) in the theory, $g_i$ is the number of real degrees of freedom of state $i$.
*Hint: a Weyl fermion (plus antiparticle) have $g_f = 2$, a Dirac fermion has $g_f = 4$, a massless gauge boson has $g_b = 2$, a massive gauge boson has $g_b = 3$, a complex scalar has $g_b = 2$, and a real scalar has $g_b = 1$.*

f) Relative to this minimum, show that turning on a VEV for $\phi_1$ (but not $\phi_2$) does not change the potential energy. This means that $\phi_1$ is a *flat direction* of the potential. Repeat parts d) and e) for the same minimum as above but with a non-zero fixed value of $\langle\phi_1\rangle = v_1$.

**Totally Optional Extra Problem: Running Couplings**

Consider the $\lambda\phi^4$ theory discussed in class, and let us pretend that the exact value of the one-loop correction to the (1PI connected) 4-point function is

$$\Delta\Gamma^{(4)}(s) = -a\lambda^2\left[\ln\left(\frac{\Lambda^2}{s}\right) + C\right] + \delta\lambda \;,$$

where $a$ is a positive constant. This isn't quite the real answer, but it is close enough for what we want to do here.

a) Let us choose the renormalization condition for this term to be $\Delta\Gamma^{(4)}(s_0) = 0$ for the fixed reference momentum point $s_0$. Solve for $\delta\lambda$ given this condition and show that $\Delta\Gamma^{(4)}(s)$ is now finite for any value of $s$ (to one-loop).

b) The renormalization condition chosen above is not unique, and we could have chosen to set the one-loop correction to zero at a different momentum point $s_1$. In general, let us define $\lambda(s)$ to be the renormalized coupling derived from the renormalization condition $\Delta\Gamma^{(4)}(s) = 0$. How is $\lambda(s)$ related to $\lambda(s_0)$? For this, it is easiest to write $\lambda(s)$ as a function of $\lambda(s_0)$.

c) Compute $d\lambda/dt$, where $t = \ln(s/s_0)$. This defines a non-trivial differential equation for $\lambda(s)$ when the derivative is expressed in terms of $\lambda(s)$. (Note that to $\mathcal{O}(\lambda^2)$ you can just replace $\lambda(s_0)$ with $\lambda(s)$.) Solve the differential equation for $\lambda(s)$ subject to the boundary condition $\lambda(s = s_0) = \lambda(s_0)$.

d) Expand your result to $\mathcal{O}(\lambda^2(s_0))$ and show that it reproduces the result from b) at this order. However, note that the expansion parameter in this case is not $\lambda(s_0)$ but rather $\lambda(s_0)\ln(s/s_0)$. What happens to this expansion parameter for $s \gg s_0$?

e) By defining $\lambda(s)$ through the solution to the differential equation rather than directly as in part b), we have extended the range of validity of the perturbative expansion. Even so, perturbativity can still be lost if $\lambda(s)$ grows big. Using the solution from c), at what value of $s$ does $\lambda(s)$ go to infinity? This is called a *Landau pole*.

f) Suppose we had instead that $a = -|a| < 0$. Assuming that $\lambda(s_0)$ is small, what happens to $\lambda(s)$ as $s \to \infty$? What happens as $s$ becomes much smaller than $s_0$?