P528 Notes #3: Non-Abelian Gauge Theories

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We have just seen that QED has an underlying invariance under $U(1)$ gauge transformations, and this invariance determines nearly the whole structure of the theory. Each charged field in QED picks up a spacetime-dependent phase under a gauge transformation proportional to its electric charge, and this completely fixes the coupling of the field to the photon. In contrast, the photon field A_μ does not transform, aside from a shift by the derivative of the gauge transformation parameter, and it does not couple to itself.

Of course, we also know that $U(1)$ is just the tip of the iceberg when it comes to compact Lie groups. From this point of view, it is completely natural to try to construct field theories with a gauge invariance under non-Abelian transformation groups such as $SU(N)$ or its many friends. This is what we will do here. Along the way, we will see that much of the structure of QED goes through unchanged, but that there are a few very important differences. The most significant of these is that the gauge field of a non-Abelian gauge group will turn out to have interactions with itself.

1 Non-Abelian Gauge Invariance

Consider an irreducible representation (= irrep) r of a non-Abelian compact Lie group G. If the irrep has dimension n, we can write the representation matrices according to[1]

$$U_r \;=\; e^{i\alpha^a t^a_r} \;:=\; \sum_{n=0}^{\infty}\frac{1}{n!}(i\alpha^a t^a_r)^n \ , \tag{1}$$

where the generators t^a_r are $(n\times n)$ Hermitian matrices satisfying the Lie algebra relation

$$[t^a_r, t^b_r] = if^{abc}t^c \ . \tag{2}$$

This representation acts on an n-dimensional vector space.

A set of n fields is said to transform under the representation r if the gauge transformation law for them is

$$\begin{aligned}\psi_i \;&\to\; (U_r)_{ij}\psi_j = \left(e^{i\alpha^a t^a}\right)_{ij}\psi_j \\ &=\; \psi_i + i\alpha^a (t^a_r)_{ij}\psi_j + \mathcal{O}(\alpha^2) \ .\end{aligned} \tag{3}$$

For the most part, we will just write the n-components ψ_i as a single column vector ψ and suppress the indices, $\psi \to U_r\psi$, but do keep in mind that they are there.

We now have that $\psi \to U_r\psi$ and $\bar{\psi} \to \bar{\psi}U_r^\dagger$. However, the derivative of ψ, which we will need for its kinetic term, does not transform quite so nicely if the transformation matrix

[1] As usual, any function of a matrix should be thought of as a formal power series.

varies over spacetime:

$$\partial_\mu \psi \to U_r \partial_\mu \psi + (\partial_\mu U_r)\psi \ . \tag{4}$$

It follows that $\bar{\psi} i\gamma_\mu \partial_\mu \psi$ is not invariant due to the derivative of the transformation matrix. Note that we have to be a bit careful with this piece because, in contrast to the Abelian case, one typically has

$$\partial_\mu (e^{i\alpha^a t^a}) \neq i(\partial_\mu \alpha^a t_r^a) e^{i\alpha^a t_r^a} \neq e^{i\alpha^a t_r^a}\, i(\partial_\mu \alpha^a\, t_r^a) \ . \tag{5}$$

The reason for this is that $\alpha^a t_r^a$ and $\partial_\mu \alpha^a t_r^a$ do not commute with each other unless $\partial_\mu \alpha^a = \lambda \alpha^a$. Thus, we will stick with the correct expression of Eq. (4).

To make the kinetic term for the charged field invariant under non-Abelian transformations of the form of Eq. (4), we QED and introduce a matrix-valued vector field

$$A_{r\mu} \ := \ A_\mu^a t_r^a \tag{6}$$

that transforms according to

$$A_{r\mu} \ \to \ U_r A_{r\mu} U_r^{-1} + \frac{1}{ig} U_r (\partial_\mu U_r^{-1}) \ := \ \frac{1}{ig} U_r (D_\mu U_r^{-1}) \ . \tag{7}$$

Coupling this to the charged field, we get

$$\begin{aligned} (\partial_\mu + igA_{r\mu})\psi \ &\to \ \left[\mathbb{I}\,\partial_\mu + ig\left(U_r A_\mu U_r^{-1} + \frac{1}{ig} U_r \partial_\mu U_r^{-1}\right)\right] U_r \psi \\ &= \ U_r\,(\partial_\mu + igA_{r\mu})\,\psi \ . \end{aligned} \tag{8}$$

As in QED, we call the combination $D_{r\mu} = (\partial_\mu + igA_{r\mu})$ the covariant derivative operator for the representation r,[2] and $\bar{\psi} i\gamma^\mu D_{r\mu}\psi$ (for ψ a fermion) is gauge invariant.

The definitions above lead to (at least) two important questions. First, how do we construct a reasonable gauge-invariant kinetic term for the gauge field? Second, our definition of the gauge field $A_{r\mu}$ depends on the representation of the corresponding matter field, so do we need additional gauge fields for other matter fields transforming under different representations? It turns out that the answers to both questions are closely related.

Starting with the second question, working out the transformation law explicitly to linear order in the parameter α^a one finds that

$$\begin{aligned} A_\mu^a t_r^a \to A'^a_{r\mu} t_r^a \ &= \ U_r A_{r\mu} U_r^{-1} + \frac{1}{ig} U_r \partial_\mu U_r^{-1} \\ &= \ \left(A_\mu^a + f^{abc} A_\mu^b \alpha^c - \frac{1}{g}\partial_\mu \alpha^a\right) t_r^a + \mathcal{O}(\alpha^2) \ . \end{aligned} \tag{9}$$

At this order, we see that the transformation law for the coefficient fields A_μ^a is independent of the specific representation. Moreover, $\alpha^a = \mathit{constant}$, it corresponds to A_μ^a transforming

[2] The ∂_μ part of this operator is implicitly multiplied by the $n \times n$ identity matrix.

in the adjoint representation of the group. One can extend this result to all orders in α^a (by induction or by composing infinitesimal transformations). Therefore, it is sufficient to introduce a single set of coefficient gauge fields A^a_μ to ensure the invariance of the kinetic terms of fields transforming under any representation at all of the gauge group.

Moving next to the kinetic term for these gauge fields, a reasonable first guess would be to start with the combination $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu$. Unfortunately, it has a complicated gauge transformation property and it is not at all clear how to put it into a gauge-invariant kinetic term. Instead, let us use the nice gauge transformation properties of the covariant derivative as our guide. Acting on a field transforming under the rep r, we found that the covariant derivative of that field transforms as $D_\mu\psi \to U_r D_\mu\psi$. Equivalently, as a differential operator, we have that $D_\mu \to U_r D_\mu U_r^{-1}$. In the same way, the covariant commutator differential operator transforms as $[D_\mu, D_\nu] \to U_r[D_\mu, D_\nu]U_r^{-1}$. Now, working out the effect of this operator on any field ψ transforming in the rep r, one finds

$$\begin{aligned}[D_\mu, D_\nu]\psi &= ig\, t^a_r(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc}A^b_\mu A^c_\nu)\psi \\ &:= ig\, t^a_r F^a_{\mu\nu}\psi\ .\end{aligned} \tag{10}$$

Aha! The commutator of these two differential operators does not involve any derivatives of ψ at all, it transforms in a reasonable way, and it contains the pieces we want to make up a vector kinetic term.

A reasonable gauge-invariant kinetic term for the gauge fields is therefore[3]

$$\begin{aligned}\mathscr{L} &\supset -\frac{1}{4(ig)^2T_2(r)}tr([D_\mu, D_\nu][D^\mu, D^\nu]) \\ &= -\frac{1}{4(ig)^2T_2(r)}\,(ig)^2F^a_{\mu\nu}F^{b\,\mu\nu}\,tr(t^a_r t^b_r) \\ &= -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}.\end{aligned} \tag{11}$$

Note that even though we used a specific representation to define the gauge kinetic term, the third line of Eq. (11) shows that it can be written in a way that is representation-independent.

In all the discussion here, we have cobbled together a sensible gauge-invariant Lagrangian by fiddling around. However, a slightly more careful treatment shows that the matter-gauge couplings we have obtained are essentially unique. In other words, the requirement of gauge-invariance completely fixes the structure of the gauge interactions. As in QED, this is why it makes sense to think of gauge-invariance as the fundamental underlying feature. Also like in QED, gauge-invariance is to be treated as an equivalence relation rather than a genuine symmetry.

2 Computing with Non-Abelian Gauge Theories

Based on the discussion above, we are now able to write down the Lagrangian for a gauge field and a set of fermions ψ transforming in a representation r of the (possibly non-Abelian)

[3]This is gauge invariant due to the cyclicity of the trace: $tr(UMU^{-1}) = tr(M)$.

gauge group G.[4] The kinetic and gauge-matter interaction terms are

$$\mathscr{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi} i\gamma^\mu D_\mu \psi + \ldots \tag{12}$$

Additional terms can include matter-matter interactions and higher-dimensional operators provided they are consistent with gauge invariance. It is also straightforward to add other fermion species transforming under different represenations of the gauge group. The form of the covariant derivative operator implicitly depends on the representation of the field upon which it acts, and is given by

$$\psi i\gamma^\mu D_\mu \psi = \bar{\psi} i\gamma^\mu(\partial_\mu + igA^a_\mu t^a_r)\psi, \tag{13}$$

where t^a_r are the generators of the representation r. Note that $t^a_r = 0$ when r is the trivial representation. In other words, a field transforming under the trivial representation of a gauge group does not couple to the gauge boson, and is said to be uncharged. Expanding out the gauge kinetic term, one obtains

$$\begin{aligned} -\frac{1}{4}(F^a_{\mu\nu})^2 &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 \\ &\quad + \frac{1}{2} g\, f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}. \end{aligned} \tag{14}$$

The first term is evidently the kinetic term for the vector, while the second two terms are gauge boson self-interactions induced by the non-Abelian nature of the gauge group; this is perhaps the most important consequence of having a non-Abelian group. These expressions also reduce to the Abelian case if we set $f^{abc} \to 0$ and $t^a_r \to Q$, where Q is the $U(1)$ charge of the field ψ.

Starting from Eq. (13) and Eq. (14) we can derive all the Feynman rules for gauge interactions in a non-Abelian gauge theory. The final result is very nearly identical to QED with some additional group theoretic factors for decoration. However, there are a few important differences that must be taken into account.

In order to obtain a sensible quantum propagator for the gauge field, it is usually necessary to choose a specific gauge.[5] A very popular family of gauge choices goes by the name of R_ξ, with each choice in the family characterized by a constant parameter ξ. This leads to a vector propagator for $A^a_\mu \to A^b_\nu$ of

$$D^{ab}_{\mu\nu}(p) = \frac{i}{p^2}\delta^{ab}\left[-\eta_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]. \tag{15}$$

The corresponding Feynman rule is shown in Fig. 1. Some popular ξ values are the Landau gauge with $\xi = 0$, and the Feynman-'t Hooft gauge with $\xi = 1$. Any observable quantity must be independent of ξ due to the requirement of gauge invariance. This can be a useful way to check complicated calculations.

[4] In the homework you will learn how to add charged scalars to the theory.

[5] The same is true for QED, but there we can get away with ignoring the implications when computing Feynman diagrams.

A full quantum derivation of the R_ξ-gauge propagator of Eq. (15) also leads one to include an additional set of massless *Faddeev-Popov ghost* fields transforming under the adjoint rep of the gauge group. Ghost fields have the unusual property of being anti-commuting Lorentz scalars. (Typically, even-spin fields (bosons) are commuting while odd-spin fields (fermions) are anti-commuting.) The interpretation is that the ghosts do not represent physical particle excitations.[6] Instead, they play the role of "negative degrees of freedom" in Feynman diagram calculations to cancel off the gauge redundancy of the vector gauge fields. In practice, this means that ghost fields only appear as intermediate states in loop diagrams, and never appear as on-shell external states in a physical process. With one additional minor requirement to be discussed below, this implies that we can completely ignore the ghost fields as long as we stick to tree-level processes.

The propagator of a fermion field is nearly identical to the QED case. For $\psi_i \to \psi_j$ (where i and j are the indices of the rep), we have

$$D_{ij}(p) = \delta_{ij}\frac{i(\not{p}+m)}{p^2-m^2}\,. \tag{16}$$

Similarly, for a charged complex scalar $\phi_i \to \phi_j$,

$$D_{ij}(p) = \delta_{ij}\frac{i}{p^2-m^2}\,. \tag{17}$$

We illustrate the corresponding diagrams in Fig. 1.

Vertex factors are straightforward to derive from Eqs. (13,14). The vertex corresponding to the fermion-vector $\psi_j \to \psi_i A^a_\mu$ interaction is

$$V_{ffG} = -ig\,(t^a_r)_{ij}\gamma_\mu. \tag{18}$$

Notice how the indices on the generator matrix match up with the representation indices on the fermions. There are also three- and four-point gauge self interactions. For $A^a_\mu A^b_\nu A^c_\rho$ we have

$$V_{3G} = -g\,f^{abc}\left[\eta_{\mu\nu}(p_a-p_b)_\rho + \eta_{\nu\rho}(p_b-p_c)_\mu + \eta_{\rho\mu}(p_c-p_a)_\nu\right], \tag{19}$$

where $p_{a,b,c}$ are the *incoming* momenta carried by the vectors $A^{a,b,c}_{\mu,\nu,\rho}$ at the vertex (see Fig. 1). For the $A^a_\mu A^b_\nu A^c_\rho A^d_\sigma$ vertex we get

$$\begin{aligned} V_{4G} &= -ig^2[f^{abe}f^{cde}(\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\sigma}\eta_{\nu\rho}) \\ &\qquad + f^{ace}f^{bde}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\sigma}\eta_{\nu\rho}) \\ &\qquad + f^{ade}f^{bce}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\rho}\eta_{\nu\sigma})]. \end{aligned} \tag{20}$$

Again, take a look at Fig. 1.

Feynman diagram calculations in non-Abelian gauge theories are very similar to those in QED up to some additional group theoretic factors and the gauge field self-interactions.

[6] Ghosts also come up in QED and other Abelian gauge theories, but since the adjoint rep of such theories is trivial, they do not couple to anything and can be ignored when computing Feynman diagrams.

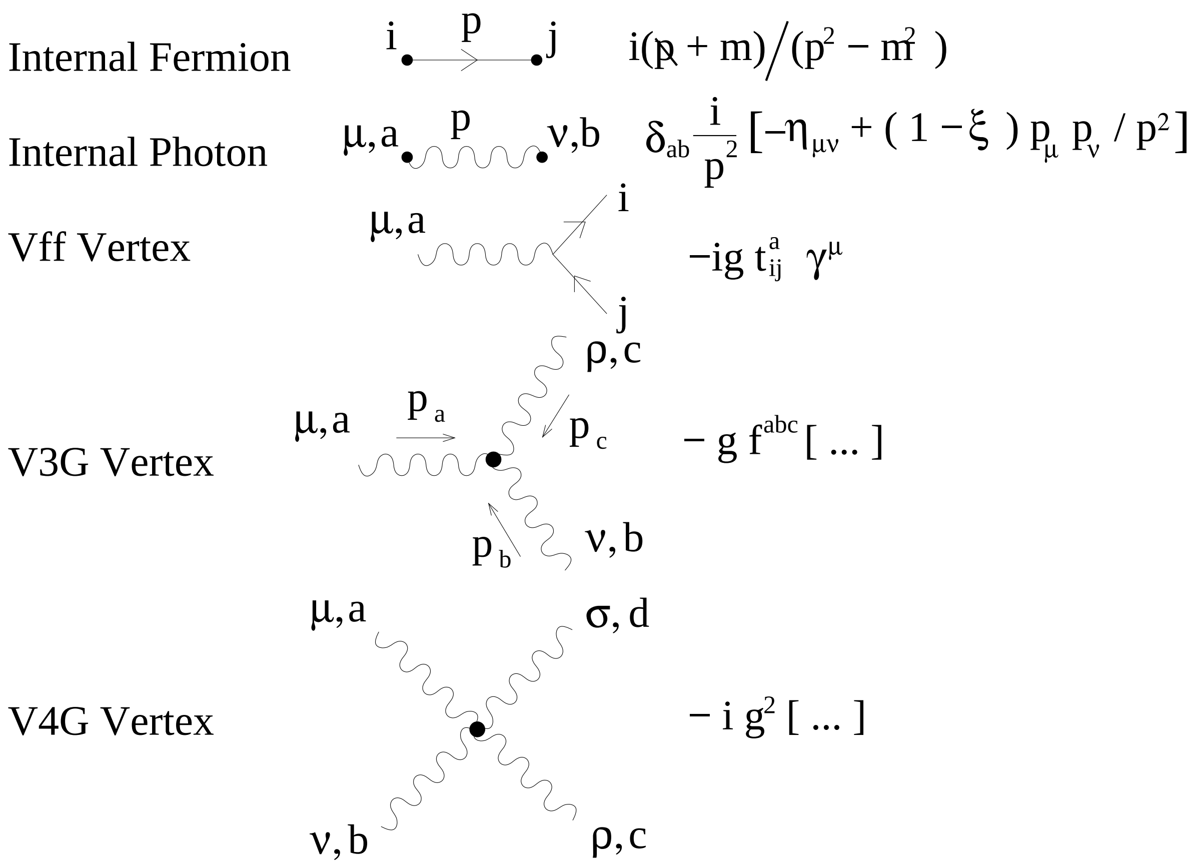

Figure 1: Feynman rules for a non-Abelian gauge theory.

As in QED, one builds up an amplitude by writing down all the Feynman diagrams for the process. Each diagram has a numerical value which is built up by tracing backwards along fermion lines, putting in internal propagators and vertex factors, and adding the initial- and final-state polarization vectors and fermion spinors. The main complication is that one must also keep track of non-Abelian group theory factors. For example, each gauge boson line has an adjoint index associated to it while each charged fermion or scalar line transforming in the rep r has an index corresponding that rep.

The amplitude that is computed has specific external spin states, vector polarizations, and group theory values. In most cases we want unpolarized cross-sections that are summed over all distinct final states and averaged over all distinct initial states. The fermion spin and vector polarization parts are nearly identical to QED, but now we also have to sum over the distinct states in a given rep. For example, the amplitude for a process involving $\psi_j + X \to \psi_i + Y$ will take the form $\mathcal{M}_{ij}$, where i and j are indices for the rep of ψ. The squared and summed matrix element for the process will then involve

$$\text{``}|\mathcal{M}|^2\text{''} = \frac{1}{d(r)} \sum_{i,j} \mathcal{M}^*_{ij} \mathcal{M}_{ij}, \tag{21}$$

where $d(r)$ is the dimension of the rep of ψ.

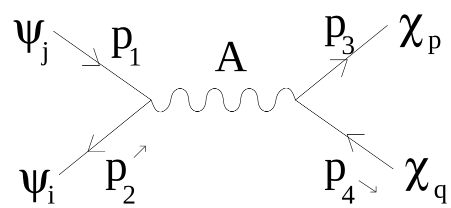

Figure 2: The leading Feynman diagram for $\psi\bar{\psi} \to \chi\bar{\chi}$.

e.g. 1 $\psi\bar{\psi} \to \chi\bar{\chi}$

Suppose ψ and χ are massless fermions transforming under the reps r_ψ and r_χ of the non-Abelian gauge group G. The leading Feynman diagram for this process is given in Fig. 2. The amplitude is

$$-i\mathcal{M} = -ig^2(t^a_{r_\psi})_{ij}(t^b_{r_\chi})_{pq}\frac{\delta_{ab}}{p^2}(\bar{u}_3\gamma^\mu v_4)(\bar{v}_2\gamma^\nu u_1)\left[-\eta_{\mu\nu} + (1-\xi)p_\mu p_\nu/p^2\right]. \tag{22}$$

Here, $p = (p_1 + p_2) = (p_3 + p_4)$, and the subscripts label the momenta of the spinors (with spinor indices contracted). Squaring and summing/averaging the matrix element, the spin part comes out just like in the $(e^+e^- \to \mu^+\mu^-)$ example we looked into previously in QED (and one finds that the ξ-dependent part does not contribute in the end). There is, however, a new group theory piece. The net result is

$$\text{``}|\mathcal{M}|^{2\prime\prime} = (GT)\,\frac{8g^4}{p^2}[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)] \tag{23}$$

with the group theory factor given by

$$\begin{aligned}(GT) &= \frac{1}{d^2(r_\psi)}\sum_{i,j}\sum_{p,q}(t^a_{r_\psi})^*_{ij}(t^c_{r_\psi})_{ij}(t^b_{r_\chi})^*_{pq}(t^d_{r_\chi})_{pq}\,\delta_{ab}\delta_{cd}\\ &= \frac{1}{d^2(r_\psi)}\,tr(t^a_{r_\psi}t^c_{r_\psi})\,tr(t^a_{r_\chi}t^c_{r_\chi})\,\delta_{ab}\delta_{cd}\\ &= \frac{d(A)}{d^2(r_\psi)}T_2(r_\psi)T_2(r_\chi).\end{aligned} \tag{24}$$

In the second line we have made use of the Hermiticity of the t^a to write $(t^a)^* = (t^a)^t$ while in the third line we have used $\delta^{ac}\delta^{bd}\delta_{ab}\delta_{cd} = \delta^c{}_c = d(A)$. A useful trick for obtaining the gauge matrix factors is to trace backwards along the "gauge flow" in the diagram, much like one traces backwards along fermion lines to get the spinor factors.

Relative to QED (or other purely Abelian gauge theories), there is one additional complication related to the polarization of external vector states. Recall that in QED, we were able to use a polarization completeness relation to simplify the polarization sums:

$$\sum_\lambda \epsilon^*_\mu(p,\lambda)\epsilon_\nu(p,\lambda) = -\eta_{\mu\nu} + (\text{stuff you can ignore}) \qquad (\text{Abelian case}). \tag{25}$$

The "extra stuff" here is related to the fact that there are only two distinct physical polarizations for a massless vector, whereas four states would be needed for full completeness, corresponding to a sum that produces $\eta_{\mu\nu}$ alone. Fortunately, in Abelian gauge theories the extra stuff always vanishes automatically when it is contracted with a physical amplitude and can therefore be neglected. In the non-Abelian case it turns out that you can't always get away with ignoring the extra stuff. There are various ways of handling this, but in many cases the easiest is to specify explicitly the two transverse polarization vectors $\epsilon^\mu(\vec{p},\lambda)$, $\lambda = 1, 2$, and sum over them. You can choose these however you want provided they satisfy the conditions

$$\epsilon^*(\vec{p},\lambda)\cdot\epsilon(\vec{p},\lambda') = -\delta_{\lambda\lambda'}, \qquad (1,\vec{0})\cdot\epsilon(p,\lambda) = 0, \qquad p\cdot\epsilon(\vec{p},\lambda) = 0. \tag{26}$$

For example, if $\vec{p} = p\hat{z}$, two popular choices are $\{(0,1,0,0),\,(0,0,1,0)\}$ (linear polarizations) and $\{(0,1,i,0)/\sqrt{2},\,(0,1,-i,0)/\sqrt{2}\}$ (right- and left-handed circular polarizations).

3 The Fundamental QCD Lagrangian

Quantum chromodynamics is the underlying theory of the strong force. It is a non-Abelian gauge theory with gauge group $SU(3)$. The gauge fields (of which there are $8 = 3^2 - 1$ components) are called gluons G^a_μ. In addition, there are six fermionic quark fields, $q =$ u, d, c, s, t, b, each transforming under the fundamental **3** representation of $SU(3)$. The different quark fields are called *flavours*. For each flavour, the three components of the **3** representation are called colours: $q = q_i$, $i = 1, 2, 3$. From the point of view of QCD, there is nothing terribly fundamental about flavour while the colours are an essential part of the underlying gauge symmetry structure. The terminology of flavour and colour is also frequently applied to other non-Abelian gauge theories.

Given what we know about non-Abelian gauge theories, we can write down the QCD Lagrangian immediately:

$$\mathscr{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + \sum_{q=u,d,c,s,t,b} \bar{q}(i\gamma^\mu D_\mu - m_q)q, \tag{27}$$

where $D_\mu = (\partial_\mu + ig_s t^a A_\mu)$ and m_q is the mass of quark q. That's it!

References

[1] C. P. Burgess and G. D. Moore, "The standard model: A primer," *Cambridge, UK: Cambridge Univ. Pr. (2007) 542 p*

[2] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[3] S. Pokorski, "Gauge Field Theories," Cambridge, Uk: Univ. Pr. (1987) 394 P. (Cambridge Monographs On Mathematical Physics).