PHYS 528 Homework #5

Due: Feb.16, 2017

1. Fermion fun.

   a) Define $\epsilon = i\sigma^2$ and $\bar{\epsilon} = -\epsilon$. Show that $\epsilon^2 = -\mathbb{I}$, $\bar{\epsilon} = \epsilon^{-1}$, and $(\sigma^i)^*\epsilon = -\epsilon\sigma^i$.

   b) Use these results to prove that $M^t(\alpha^i)\epsilon = \epsilon M(-\alpha^i) = \epsilon M^{-1}(\alpha^i)$, for the matrix $M(\alpha^i) = e^{-i\alpha^i\sigma^i/2}$ with $\alpha^i = (\theta^i + i\beta^i)$.
   Hint: $(\sigma^i)^t = (\sigma^i)^*$, $\epsilon\epsilon = -\mathbb{I}$, *and look at the n-th term in the exponential.*

   c) Apply this to prove that for any left-handed 2-component fermion χ, the quantity $\bar{\chi} = \epsilon(\chi)^*$ transforms like a right-handed 2-component fermion. Similarly, given a right-handed fermion $\bar{\xi}$, show that $\xi = \bar{\epsilon}(\bar{\xi})^*$ transforms like a left-handed fermion.

   d) Prove that the combination $\bar{\xi}^\dagger\chi$ is Lorentz invariant. Also, rewrite this combination in terms of left-handed fermions only.

2. Dirac and Majorana fermions.

   a) The electron in QED is a Dirac fermion consisting of independent e_L and e_R LH and RH 2-component pieces (and their conjugates). How do these transform under $U(1)_{em}$ gauge transformations? Rewrite both as 2-component LH fermion fields (and their conjugates). How do these two independent LH fermion fields transform under $U(1)_{em}$?

   b) A 4-component fermion ψ is said to be *Majorana* if its two 2-component pieces are derived from the same 2-component LH fermion. Show that this implies that (up to a possible phase)

   $$\psi^c = \psi \ , \qquad \text{where} \qquad \psi^c = i\gamma^2\gamma^0(\bar{\psi})^t \ .$$

   c) A *Majorana mass* for a general 4-component fermion ψ is defined to be

   $$\mathscr{L} \supset -\frac{1}{2}M\,\overline{(\psi^c)}\psi \ .$$

   Express this operator in terms of the 2-component pieces of ψ.
   Note: do not assume that ψ *is a Majorana fermion.*

   d) Prove that a Majorana mass is forbidden if the theory has a $U(1)$ symmetry under which ψ transforms non-trivially.

   e) A pair of LH 2-component fermions χ and ξ are said to be *vector-like* if they transform under conjugate unitary representations of the underlying (non-Lorentz) symmetry group of the theory relative to each other. (*i.e.* $\chi \to e^{i\beta^a t^a_r}\chi$, $\xi \to e^{-i\beta^a t^{a*}_r}\xi$ with Hermitian t^a_r, real β^a.) Show that this allows a standard Dirac mass term $m\overline{\psi}\psi$ for the 4-component fermion $\psi = (\chi, \bar{\xi})$.

3. Compute the decay width of the Higgs boson to $b\bar{b}$. Compare this to decay rate to $c\bar{c}$ and $\tau\bar{\tau}$. Find the numerical values of these widths (in MeV) for a Higgs mass of $m_h = 125$ GeV. You may neglect the fermion masses in the kinematics, but not in the couplings.
Hint: you can look up the masses of these fermions at http://pdglive.lbl.gov/.

4. Decays of the top quark.

   a) The top quark decays nearly all the time via $t \to W^+ + b$. Based on the size of the coupling of the top to the W^+, make an educated guess for the decay width.

   b) Compute the decay width explicitly, and compare it to the measured value. You may treat the b quark as being massless to make your life easier.

   c) What happens to the top decay width as m_t becomes much larger than m_W? Compare this to the rough estimate in part a).