# PHYS 528 Homework #4

Due: Feb.9, 2017

1. $AA \to \psi\bar{\psi}$ in a general non-Abelian gauge theory with $\psi$ transforming in the rep $r$.

   a) There are two Feynman diagrams for this process: one with the vector in the s-channel and one with the fermion in the $t$-channel. Find the contribution to the amplitude for $A^a_\mu A^b_\nu \to \psi_i\bar{\psi}_j$ from the $s$-channel diagram alone.
   *Hint: the three-point vector interaction is defined for ingoing momenta on all legs. For an outgoing momentum on a leg, just swap $p \to -p$ on that leg.*

   b) Square this contribution and sum it over all final states and average over initial states (including spin and group), working in the centre-of-mass (CM) frame.
   *Hint: in the CM frame with vector momenta $p_1$ and $p_2$, $(p_1 \cdot \epsilon_2) = 0 = (p_2 \cdot \epsilon_1)$. Also, $(p_1 - p_2) \cdot (p_1 + p_2) = 0$ for massless vectors. Use this to simplify the amplitude enormously before squaring.*

   c) Write down the contribution to the amplitude $A^a_\mu A^b_\nu \to \psi_i\bar{\psi}_j$ from the $t$-channel diagram alone.

   d) Work out the group theory factor corresponding to the $t$-channel diagram when one squares this contribution and sums/averages it over all final/initial states.

2. Scalar expansions.

   Consider a general theory of $n$ real scalars $\phi_i$ $(i = 1, \ldots, n)$ with Lagrangian

$$\mathscr{L} = \frac{1}{2}\sum_{i=1}^{n}(\partial\phi_i)^2 - V(\phi_i)\ . \tag{1}$$

   If $\{\phi_i = \langle\phi_i\rangle\}$ is a minimum of the potential, show that the fields $h_i(x) = \phi_i(x) - \langle\phi_i\rangle$ have canonical normalization and vanish at the minimum of the potential. Next, expand the potential about the minimum in a power series in $h_i$ and show that $\partial^2 V/\partial\phi_i\partial\phi_j|_{\langle\phi\rangle}$ is the mass matrix for the scalars $h_i$.

3. SSB and NGBs.

   a) In the spontaneously broken global $U(1)$ theory discussed in the notes, work out the kinetic term and potential in terms of the new polar field variables we defined.

   b) Suppose we have the same theory but with $-\mu^2 \to +\mu^2$ in the potential. Is there still spontaneous symmetry breaking? What are the particle masses of all the real scalar degrees of freedom? And in this case is it still useful (for doing perturbation theory) to expand the complex scalar as $\phi(x) = e^{iA(x)}B(x)$ with $A(x)$ and $B(x)$ real scalars?
   *Hint: we really want canonical kinetic terms!*

c) For the same theory, expanded around the vacuum $\langle\phi\rangle = e^{i\beta}\, v$, work out the mass matrix $\partial^2 V/\partial\phi_i\partial\phi_j|$ in terms of the original field variables $\phi$ and $\phi^*$ and show that it has a zero determinant. Also, find $F_i^a(\langle\phi\rangle)$ for an infinitesimal phase rotation and show that it is a zero eigenvalue of this mass matrix.
*Hint: treat $\phi$ and $\phi^*$ as independent degrees of freedom.*

d) For the global $SU(2)\times U(1)$-symmetric theory discussed in the notes, work out the full Lagrangian in terms of the new field variables

$$\begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} = \begin{pmatrix} (\phi_{+r} + i\phi_{+i})/\sqrt{2} \\ v + (\phi_{0r} + i\phi_{0i})/\sqrt{2} \end{pmatrix} \tag{2}$$

This choice corresponds to an expansion about the vacuum with $\alpha^a = \beta = 0$. What are the mass eigenvalues?

4. A semi-realistic Higgs.

a) Starting from the global $SU(2)\times U(1)$-symmetric theory discussed in the notes, elevate this to a theory that is invariant under local $SU(2)\times U(1)$ transformations by adding an appropriate set of vector gauge fields and couplings. What is the corresponding Lagrangian?
*Hint: each gauge factor has its own gauge field and its own gauge coupling.*

b) Work out the commutation relations of the modified set of generators $\tilde{t}$ and $\{t^B_{G/H}\}$ discussed in the notes.

c) Expand this theory around the vacuum after making a nice choice of gauge. Find the masses of all the physical scalar and vector particles in the theory, and check that the numbers of degrees of freedom match up.