# PHYS 528 Lecture Notes #4

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## 1 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking (SSB) is a very simple idea that has very profound consequences in QFT. The main idea is that the underlying theory has a symmetry while the underlying vacuum state does not. The way this works is seen most easily in the following example.

Consider the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi), \tag{1}$$

with the potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4. \tag{2}$$

This theory obviously has a discrete symmetry under $\phi \to -\phi$. However, we also see that the quadratic term does not have the right sign to be a scalar mass term, unless we interpret the mass as $m = i\sqrt{\mu}$. Something is clearly wrong. The way to resolve this can be found by looking at the shape of the potential, which we illustrate in Fig. 1. Evidently the origin of the field space, $\phi = 0$, is not a stable minimum of the potential. For example, solving the classical equation of motion for a scalar starting at $\phi(t = t_0) = 0$, one finds that the amplitude initially grows exponentially (for $\partial_t\phi(t = t_0) \neq 0$). Instead, the stable minima lie at

$$\langle\phi\rangle = \pm\mu/\sqrt{\lambda} \equiv \pm\, v. \tag{3}$$

In the quantum version of the field theory, the rule of thumb is to choose a specific stable local minimum and expand perturbatively in small fluctuations around it. Let's choose the positive solution above and write

$$\phi(x) = v + h(x), \tag{4}$$

where $h(x)$ is also a real scalar field. Plugging this form into the original Lagrangian, we see that the kinetic term for $h(x)$ is canonical while the potential becomes

$$V = -\frac{1}{4}\lambda v^2 + \frac{1}{2}\lambda\, v^2\, h^2 + \lambda\, v\, h^3 + \frac{\lambda}{4}h^4. \tag{5}$$

This potential has a stable minimum at $h = 0$, a sensible mass term for $h$ of $m_h = \sqrt{\lambda}v$, and some $h$ self-interactions. On the other hand, there is no $h \to -h$ symmetry (unless we also swap $v \to -v$). As a result, we say that the symmetry has been *spontaneously broken.*

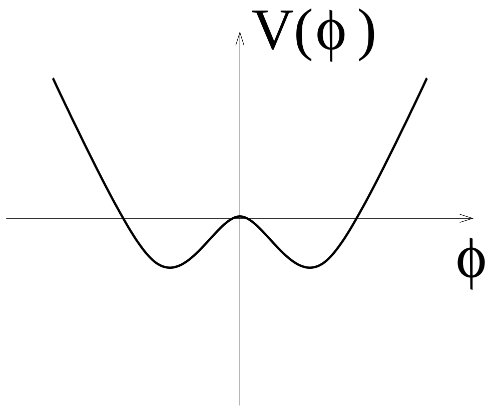


Figure 1: Feynman rules for a non-Abelian gauge theory.

At this point you might be wondering why we needed to choose a single vacuum. In ordinary one-particle quantum mechanics, the true ground state for a potential with two equally-deep minima, $|+\rangle$ and $|-\rangle$ say, is a linear combination of the two: $|0\rangle = (|+\rangle + |-\rangle)/\sqrt{2})$. In fact, under the reflection symmetry $|+\rangle \leftrightarrow |-\rangle$, implying that this linear combination is symmetric under the symmetry. The energy of this state is also lower than either of the $|+\rangle$ or $|-\rangle$ states. If ones starts in either one of these states, there is a finite probability to tunnel to the other state and the system can eventually settle down to the true ground state.

This is not the case for a quantum field theory. The essential difference is that the QFT we are working with lives in an infinite volume. Starting with $\phi(x) = +v$, the energy needed to go to $\phi(x) = -v$ is proportional to the volume and is therefore infinite. (The energy cost in one-particle QM is finite.) This implies that it is not possible to tunnel from one vacuum to the other in the field theory in a finite amount of time.[1] As a result, we need to choose a single specific vacuum state to expand around in the QFT case. Since the two vacua here are physically distinct and separated by an infinite energy cost, expanding about one or the other represents a distinct physical theory. In other words, our QFT is defined both by the Lagrangian of Eq. (1) together with the choice of vacuum state $|+\rangle$ ($\langle\phi\rangle = +v$) or $|-\rangle$ ($\langle\phi\rangle = -v$).

When we discussed symmetries earlier, we saw that continuous symmetries are particularly interesting and lead to conserved quantities. Spontaneously breaking a continuous symmetry provides even more surprises. The simplest example of this is given by the $U(1)$-symmetric Lagrangian

$$\mathscr{L} = |\partial\phi|^2 - V(\phi) \tag{6}$$

[1]Even when the volume is finite, we should still work with a single vacuum when the tunnelling time is much longer than all the other relevant time scales in the system.

with

$$V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4. \tag{7}$$

This potential obviously has a global $U(1)$ symmetry (as does the kinetic term) and is sometimes called a "wine bottle" or a "Mexican hat", and looks just like that in Fig. 1 after rotating the profile around the vertical axis normal to the $Re(\phi)$-$Im(\phi)$ plane. The stable minima are all those that satisfy the condition

$$|\phi|^2 = \mu^2/\lambda \equiv v^2. \tag{8}$$

Thus, the set of vacuum states is given by

$$\langle\phi\rangle = e^{i\beta}v \quad \leftrightarrow \quad |\beta\rangle. \tag{9}$$

Put another way, we have a circle's worth of distinct vacuum states that we can label by the parameter $\beta \in [0, 2\pi)$. Note that even though these vacua do not have an energy barrier separating them, there is still an infinite gradient energy cost to go from one vacuum state to another, so we must still pick a specific vacuum to expand around. Any such vacuum breaks the $U(1)$ invariance since $|\beta\rangle \to |\beta + \alpha\rangle$ under $\phi \to e^{i\alpha}\phi$.

Choosing the vacuum state $|\beta\rangle$, we can expand about it by changing our field variables to a polar form:

$$\phi = (v + h(x)/\sqrt{2})e^{i(\beta+\rho(x)/\sqrt{2}v)}. \tag{10}$$

Note that we have just exchanged our independent real degrees of freedom from $Re(\phi)$ and $Im(\phi)$ to $h(x)$ and $\rho(x)$. Both $h(x)$ and $\rho(x)$ vanish in the vacuum. In general, you can choose any set of field variables you like as long as they lead to a sensible set of kinetic and mass terms, although a judicious choice can save you a lot of unneeded work. Plugging these new variables into the Lagrangian, we get

$$|\partial\phi|^2 = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(1 + h/\sqrt{2}v)(\partial\rho)^2, \tag{11}$$

as well as

$$V(\phi) = (const) + \frac{1}{2}(2\lambda v^2)h^2 + \frac{\lambda}{\sqrt{2}}vh^3 + \frac{\lambda}{8}h^4. \tag{12}$$

This gives canonical kinetic terms for both $h$ and $\rho$, some interactions, and masses of $m_h = \sqrt{2\lambda}\,v$ and $m_\rho = 0$.

The masslessness of $\rho(x)$ here is not an accident. Under $U(1)$ transformations there is still a hidden symmetry under which $h$ is invariant and

$$\rho/\sqrt{2}v \to \rho/\sqrt{2}v + \alpha. \tag{13}$$

In other words, the $U(1)$ has become a shift symmetry for $\rho$. This symmetry forbids non-derivative interactions involving $\rho$, and thus allows no mass term for this field. It turns out that this is a generic feature of spontaneously broken continuous symmetries, and the corresponding massless states are called *Nambu-Goldstone Bosons* (NGBs). Such symmetries are sometimes said to be realized *non-linearly.*

This statement about NGBs can be proved very generally. Suppose we have a theory whose Lagrangian is invariant under the continuous symmetry group $G$, but whose vacuum state is only invariant under a smaller subgroup $H$. Under an infinitessimal $G$ transformation we have

$$\phi_i \to \phi_i + \delta\alpha^a F_i^a(\phi). \tag{14}$$

Invariance of the potential under arbitrary $G$ transformations implies that $V(\phi + \delta\alpha^a F^a) = V(\phi)$, which translates into the condition

$$0 = F_i^a \frac{\partial V}{\partial \phi_i}, \qquad a = 1, \ldots, d(G). \tag{15}$$

Taking this relation and differentiating with respect to $\phi_j$ and evaluating at the minimum of the potential, we get

$$0 = \left.\frac{\partial F_i^a}{\partial \phi_j}\frac{\partial V}{\partial \phi_i}\right|_0 + \left.F_i^a \frac{\partial^2 V}{\partial \phi_j \partial \phi_i}\right|_0, \tag{16}$$

where $(...)|_0$ implies that one should evaluate the fields at the minimum, $\phi = \langle\phi\rangle$. The first term vanishes at the minimum of the potential, while the second derivative in the second term corresponds to the scalar mass matrix of the theory, $m^2_{ij} = \partial^a V/\partial\phi_i\partial\phi_j|_0$. Now, invariance of the vacuum under some $G$ transformations implies that

$$F^a(\phi = \langle\phi\rangle) = 0 \qquad \Longleftrightarrow \qquad \text{the } a\text{-th generator leaves the vacuum invariant.} \tag{17}$$

Applying this to Eq. (16), we see that the mass matrix has a zero eigenvalue for every generator that does not leave the vacuum invariant. These zero eigenvalues are precisely the massless NGBs of the theory.

It is easy to count the number of Goldstone modes in a more organized way. We can choose generators $\{p^a, q^b\}$ for $G$ such that the $p^a$ generate the $H$ subgroup that leaves the vacuum invariant, and the $q^b$ make up the rest. Sometimes it is said that the $q^b$ generate the so-called coset space $G/H$, which may or may not be a subgroup of $G$. The indices of the $p^a$ run over $a = 1, 2, \ldots, d(H)$, and those of the $q^b$ run from $b = d(H)+1, \ldots, d(G)$. Our result above shows that the Goldstone bosons correspond in a one-to-one way with the generators $q^b$ of $G/H$:

$$\text{NGB} \leftrightarrow \text{generator of } G/H. \tag{18}$$

There are precisely $[d(G) - d(H)]$ of them.

A slightly more complicated example of NGBs is given by the theory with Lagrangian

$$\mathscr{L} = (\partial\phi)^\dagger(\partial\phi) - V(\phi), \tag{19}$$

where

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \frac{\lambda}{2}(\phi^\dagger \phi)^2, \tag{20}$$

and

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \tag{21}$$

is a complex scalar doublet. This theory is invariant under global $SU(2) \times U(1)$ transformations. The minimum of the theory is defined by

$$\phi^\dagger \phi = \mu^2/\lambda \equiv v^2. \tag{22}$$

Thus, the most general vacuum state can be written in the form

$$\langle \phi \rangle = e^{i\beta Q} e^{i\alpha^a t^a} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{23}$$

where the $t^a = \sigma^a/2$ generate $SU(2)$ and $Q$ is the $U(1)$ charge of $\phi$.

Let's choose the vacuum state corresponding to $\beta = 0 = \alpha^a$. None of the four $SU(2) \times U(1)$ generators leave this vacuum invariant individually, but there is a single linear combination that does:

$$\tilde{t} = \frac{1}{2}\mathbb{I} + t^3. \tag{24}$$

This generates a $U(1)$ subgroup of $SU(2) \times U(1)$ under which $\phi_+$ has charge $\tilde{Q}_+ = 1/2 + 1/2 = 1$ and $\phi_0$ has charge $\tilde{Q}_0 = q/2 - 1/2 = 0$, and corresponds to the $p^a$ discussed above. We can choose as generators for the rest of the group

$$\{t'^a\} = \left\{ (t^1 + it^2)/\sqrt{2},\, (t^1 - it^2)/\sqrt{2},\, (-\frac{1}{2}\mathbb{I} + t^3) \right\}, \tag{25}$$

which correspond to the $q^a$.

There are lots of ways to choose new field variables that will lead to a sensible free field theory. These choices will result in identical masses but different perturbative couplings. However, at the end of the day, they should all give the same answer for physical observables (although some choices may be much easier to compute with). The choice we'll make for now is

$$\phi(x) = e^{i\rho^a(x) t'^a / f} \begin{pmatrix} 0 \\ v + h(x)/\sqrt{2} \end{pmatrix}, \tag{26}$$

where $f \sim v$ is a dimensionful constant that we'll choose a bit later. Note that there are still four real degrees of freedom, the same number that we started off with. In this form, it is clear that the potential depends only on $h(x)$ since all the factors of $\rho^a(x)$ cancel out. For the kinetic term, we get

$$(\partial\phi^\dagger)(\partial\phi) \;=\; \frac{1}{2}(\partial h)^2 + (v+h/\sqrt{2})^2\left[\partial\rho^a\partial\rho^b(t^a t^b)/f + \ldots\right]_{22}. \tag{27}$$

All the terms involving $\rho^a$ involve derivatives, and therefore there is no mass term for these fields. They are evidently the Goldstone bosons of the theory, and they match up precisely with the set of broken generators. Under infinitesimal $G$ transformations, one also sees that the $\rho^a$ transform by a shift, another tell-tale feature of NGBs.

# 2 Spontaneously "Broken" Gauge "Symmetries"

Having investigated the spontaneous breakdown of continuous global symmetries, it is natural to do the same for scalar theories with a gauge invariance. The most simple example has a single complex scalar and a $U(1)$ invariance:

$$\mathscr{L} = |(\partial_\mu + igQA_\mu)\phi|^2 - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{28}$$

with the same potential as before:

$$V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4. \tag{29}$$

The space of vacua again has $\langle\phi\rangle = ve^{i\beta}$. Choosing a fixed value of $\beta$, we can expand around this vacuum by rewriting the complex scalar as

$$\phi(x) = e^{i(\beta+\rho(x)/\sqrt{2}v)}(v+h/\sqrt{2}). \tag{30}$$

The main difference in the present case relative to the global $U(1)$ theory is that we now have the freedom to change the phase of $\phi$ by an amount that depends on spacetime. In particular, we can make a gauge transformation such that $\beta(x)+\rho(x)/\sqrt{2}v \to 0$ everywhere, or equivalently, $\phi(x) \to (v+h(x)/\sqrt{2})$. Since field configurations related by gauge transformations are physically equivalent, we can choose to work in such a gauge without changing anything that matters.

Expanding the theory in these new variables with this choice of gauge, we find

$$\begin{aligned} |D\phi|^2 &= |(\partial_\mu + igQA_\mu)\phi|^2 \\ &= \frac{1}{2}(\partial h)^2 + \frac{1}{2}(2g^2Q^2v^2)\,A_\mu A^\mu\,(v+h/\sqrt{2})^2. \end{aligned} \tag{31}$$

At this point things are looking a bit funny. The second term in Eq. (31) contains a mass term for the gauge boson with $m_A = \sqrt{2}gQv$. We are also missing any sort of NGB mode.

To see what has happened, compare the numbers of degrees of freedom (dofs) for $\langle \phi \rangle = 0$ and $\langle \phi \rangle \neq 0$. We have:

$$\langle \phi \rangle = 0 : \quad \begin{cases} \phi \text{ has 2 real dofs,} \\ A_\mu \text{ (=massless) has 2 independent polarizations} \end{cases}$$

$$\langle \phi \rangle \neq 0 : \quad \begin{cases} \phi \to h \text{ has 1 real dofs,} \\ A_\mu \text{ (=massive) has 3 independent polarizations} \end{cases}$$

Aha! $2 + 2 = 1 + 3$. The numbers of degrees of freedom match up in both cases. What has happened is that the would-be NGB mode of $\phi$ has gone to become the longitudinal polarization of the now-massive gauge boson. The highly technical term for this is that the NGB has been *eaten* by the gauge vector to give it mass. This effect is also called the Higgs mechanism, and the remaining physical scalar is called the Higgs boson.[2]

# References


[1] C. P. Burgess and G. D. Moore, "The standard model: A primer," *Cambridge, UK: Cambridge Univ. Pr. (2007) 542 p*

[2] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[3] S. Pokorski, "Gauge Field Theories," Cambridge, Uk: Univ. Pr. ( 1987) 394 P. ( Cambridge Monographs On Mathematical Physics).


[2]While it's true that Higgs was one of the people to discover this, so too did Anderson, Brout, Engler, Guralnik, Hagen, Nambu, and possibly a few others. Somehow it was Higgs' name that stuck.