PHYS 526 Notes #6: Fun with Fermions

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Having established how fields transform under Poincaré, we turn next to the simplest non-trivial representations. These are the two-compenent representations $(j_A, j_B) = (1/2, 0),\ (0, 1/2)$, that are usually referred to as left- and right-handed Weyl fermions. When we get around to quantizing the theory later , we will see that they do indeed correspond to fermionic particles. We will also study the larger reducible representation $(1/2, 0) \oplus (0, 1/2)$, which corresponds to a four-component object called a Dirac fermion. For more details, see Refs. [1, 2, 3, 4, 5, 6].

1 Weyl Fermions

A left-handed Weyl fermion is an object transforming under the $(1/2, 0)$ representation of the Lorentz group. Similarly, a right-handed Weyl fermion is an object transforming under the $(0, 1/2)$ rep. Both are two-component objects that we call *spinors*. In describing their transformation properties we will use all sort of tricks with σ matrices. So, to begin, let's review some of these tricks. From there, we will construct explicit representation matrices and build Lorentz-invariant combinations of objects that we might someday hope to add to a Lagrangian.

1.1 Tricks with σ Matrices

The Pauli σ matrices σ^a are defined to be

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1}$$

They satisfy the handy relations

$$[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c , \tag{2}$$

$$\{\sigma^a, \sigma^b\} = 2\delta^{ab} . \tag{3}$$

The first of these relations implies that $\sigma^a/2$ form a two-dimensional representation of the Lie algebra of $SU(2)$. Taken together, Eqs. (2,3) also imply that

$$e^{i\alpha^a\sigma^a/2} = \cos(\sqrt{\alpha\cdot\alpha}/2) + i\alpha^a\sigma^a \sin(\sqrt{\alpha\cdot\alpha}/2) \tag{4}$$

To derive this, expand the exponential and use Eqs. (2,3) and $AB = \{A, B\} + [A, B]$ for any operators A and B.

It is also obvious that the all the Pauli matrices are Hermitian, $\sigma^{a\dagger} = \sigma^a$. However, $(\sigma^a)^* = (\sigma^a)^t$ are not equal to σ^a when $a = 2$. Let us now define another matrix

$$\epsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5}$$

This matrix is useful because

$$(\sigma^a)^* \epsilon = -\epsilon \sigma^a, \qquad (\sigma^a)^t \epsilon = -\epsilon \sigma^a \ . \tag{6}$$

We call this the ϵ *trick*. The ϵ *trick* also works if we replace ϵ by $\bar{\epsilon}$ defined to be

$$\bar{\epsilon} = -\epsilon = \epsilon^t \ . \tag{7}$$

Note that $\epsilon^2 = -\mathbb{I} = \bar{\epsilon}^2$ and $\epsilon\bar{\epsilon} = \mathbb{I}$

Consider a general 2×2 matrix given by

$$M(\alpha^a) = e^{i\alpha^a \sigma^a/2} \ , \tag{8}$$

for some parameters α^a. When the parameters are real, this matrix is clearly unitary. However, when the α^a are complex, M is no longer unitary in general. Despite the loss of unitarity, we can still find the inverse of M in a clever way by using the ϵ *trick* and Eq. (4). Together, they imply that

$$M^t(\alpha^a)\epsilon = \epsilon M(-\alpha^a) = \epsilon M^{-1}(\alpha^a) \ . \tag{9}$$

This result will be useful soon.

1.2 Building the $(1/2, 0)$ Representation

Recall that we had for a field $\phi_C(x)$ transforming under a rep r of the Lorentz group

$$U(\Lambda)^\dagger \phi_C(x) U(\Lambda) = [M(\Lambda)]_C^{\ D} \phi_D(\Lambda^{-1}x) \ . \tag{10}$$

The transformation matrix M can be written as an exponential:

$$M(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu} J_r^{\mu\nu}\right) = M(\omega) \ , \tag{11}$$

where the representation matrices $J_r^{\mu\nu}$ satisfy the commutation relations of the Lorentz group. Note as well that specifying Λ is equivalent to specifying the parameters $\omega_{\mu\nu}$. In particular, given $\omega_{\mu\nu}$, the corresponding matrix is $\Lambda = \exp(i\omega_{\mu\nu} J_4^{\mu\nu})$ where we defined the generators $J_4^{\mu\nu}$ of the four-vector rep in `notes-5`.

We also defined

$$J^i = \frac{1}{2}\epsilon^{ijk} J^{jk}, \qquad K^i = J^{i0} \ . \tag{12}$$

In terms of them, we formed

$$A^i = \frac{1}{2}(J^i - iK^i), \qquad B^i = \frac{1}{2}(J^i + iK^i) \ . \tag{13}$$

The commutators of the As and Bs formed two independent sets of $SU(2)$ Lie algebras. The representations are therefore (j_A, j_B), where j_A and j_B refer to the dimension of each component.

To form the $(1/2, 0)$ rep, we use

$$A^i = \sigma^i/2, \qquad B^i = 0 \; . \tag{14}$$

In terms of these, we can solve for the matrices J^i and K^i in this rep:

$$J^i \;=\; A^i + B^i = \sigma^i/2 \; , \tag{15}$$
$$K^i \;=\; i(A^i - B^i) = i\sigma^i/2 \; . \tag{16}$$

It follows that

$$M(\omega) \;=\; \exp\left(\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}_{(1/2,0)}\right) \tag{17}$$
$$\;=\; e^{i(\theta^a + i\beta^a)\sigma^a/2} \tag{18}$$
$$\;=\; e^{i\alpha^a\sigma^a/2} \;=\; M(\alpha^a) \; , \tag{19}$$

where we have defined

$$\theta^a = 4\epsilon^{abc}\omega_{bc}, \qquad \beta^a = \omega_{a0} \; . \tag{20}$$

The 2×2 matrix $M(\omega)$ looks just like a regular $SU(2)$ transformation, but now with a set of three complex parameters $\alpha^a = (\theta^a + i\beta^a)$. The real part of α^a corresponds to a rotation about the a-th spatial axis, and the imaginary part corresponds to a boost in the a-th spatial direction. Note that specifying Λ, $\omega_{\mu\nu}$, or α^a all provide equivalent ways to define a Lorentz transformation.

The Lorentz transformation of a $(1/2, 0)$ spinor ψ_α is therefore

$$U^\dagger(\alpha^a)\psi_\alpha(x)U(\alpha^a) = [M(\alpha^a)]_\alpha^{\;\beta}\psi_\beta(\Lambda^{-1}x) \; , \tag{21}$$

where α, $\beta = 1, 2$. The use of Greek indices to label the components of the spinor ψ_α is traditional but unfortunate – make sure you don't confuse them with four-vector indices.

Given the form of $M(\alpha^a)$, we can build a Lorentz-invariant bilinear operator. Using our ϵ trick, we have

$$M^t(\alpha^a)\epsilon M(\alpha^a) = \epsilon \; . \tag{22}$$

This implies that given any two $(1/2, 0)$ spinors ψ and χ, the combination $[\chi^t\epsilon\psi]$ is Lorentz-invariant. Putting in indices, we have

$$[\chi^t\epsilon\psi] = \chi_\beta\epsilon^{\beta\alpha}\psi_\alpha = -(\epsilon^{\alpha\beta}\chi_\beta)\psi_\alpha \; . \tag{23}$$

For this reason, it is standard to define a spinor with a raised index,

$$\chi^\alpha := \epsilon^{\alpha\beta}\chi_\beta \; . \tag{24}$$

In terms of this, the Lorentz-invariant bilinear is written as

$$\chi\psi := \chi^\alpha\psi_\alpha \ . \tag{25}$$

We would also like to be able to lower the spinor index. Using $\epsilon\bar{\epsilon} = 1$, it follows that we can do this using $\bar{\epsilon}_{\alpha\beta}$:

$$\psi_\alpha = \bar{\epsilon}_{\alpha\beta}\psi^\beta \ . \tag{26}$$

The bar on $\bar{\epsilon}$ is usually not written explicitly. Instead, the standard notation has $\epsilon^{\alpha\beta}$ antisymmetric with $\epsilon^{12} = +1$, and $\epsilon_{\alpha\beta}$ also antisymmetric with $\epsilon_{12} = -1$. Thus, $\epsilon^{\alpha\lambda}\epsilon_{\lambda\beta} = \delta^\alpha_{\ \beta}$.

1.3 Building the (0,1/2) and Representation and More

Having constructed the $(1/2, 0)$ representation, the $(0, 1/2)$ rep is really easy. We have

$$A^i = 0, \qquad B^i = \sigma^i/2 \ , \tag{27}$$

which implies

$$\begin{aligned} J^i &= \sigma^i/2 & (28)\\ K^i &= -i\sigma^i/2 \ . & (29) \end{aligned}$$

A general finite element is therefore

$$\begin{aligned} \overline{M}(\alpha^a) &= e^{i(\theta^a - i\beta^a)\sigma^a/2} & (30)\\ &= e^{i\alpha^{a*}\sigma^a/2} \ , & (31) \end{aligned}$$

where θ^a and β^a are related to $\omega_{\mu\nu}$ in exactly the same way as before.

Given the similarity of this form to the $(1/2, 0)$ rep, we will use a peculiar but ultimately useful notation for the indices of a (0,1/2) spinor $\bar{\psi}(x)$ (where the bar on the field is part of its name, not some sort of conjugation operation):

$$U^\dagger(\alpha^a)\bar{\psi}(x)U(\alpha^a) = [\overline{M}(\alpha^a)]^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\psi}^{\dot{\beta}}(\Lambda^{-1}x) \ , \tag{32}$$

where the indices $\dot{\alpha}, \dot{\beta} = 1, 2$. With this transformation law, we can immediately form a Lorentz-invariant bilinear operator from a pair of $(0, 1/2)$ spinors $\bar{\chi}$ and $\bar{\psi}$ using our ϵ *trick*:

$$\bar{\chi}\bar{\psi} := \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = (\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}})\bar{\psi}^{\dot{\alpha}} \ , \tag{33}$$

where $\epsilon_{\dot{\alpha}\dot{\beta}}$ corresponds to the matrix $\bar{\epsilon}$. Similarly, we can raise indices using $\epsilon^{\dot{\alpha}\dot{\beta}}$,

$$\bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} \ . \tag{34}$$

The components of $\epsilon^{\dot{\alpha}\dot{\beta}}$ are equal to those of $\epsilon^{\alpha\beta}$, and the same for $\epsilon_{\dot{\alpha}\dot{\beta}}$ and $\epsilon_{\alpha\beta}$.

The notation we are using looks funny, but there is a good reason for it. Consider the transformation property of the $(1/2,0)$ spinor ψ with a raised index,

$$\begin{aligned}
\psi^\alpha &\rightarrow \epsilon^{\alpha\lambda}[M(\alpha^a)]_\lambda^{\ \beta}\psi_\beta && (35)\\
&= \epsilon^{\alpha\lambda}[M(\alpha^a)]_\lambda^{\ \beta}\epsilon_{\beta\kappa}\psi^\kappa && (36)\\
&= \left[e^{-i\alpha^a\sigma^{a*}/2}\right]^\alpha_{\ \kappa}\psi^\kappa\ , && (37)
\end{aligned}$$

where we have arranged the index structure of the last matrix to make it consistent. Comparing Eq. (37) to the transformation of Eq. (31), we see that $(\psi^\alpha)^*$ transforms in exactly the same way under Lorentz as a $(0,1/2)$ spinor!

Given a $(1/2,0)$ spinor ψ_α, we can therefore construct a $(0,1/2)$ spinor $\bar{\psi}^{\dot{\alpha}}$ by

$$\bar{\psi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}(\psi^*)_{\dot{\beta}}\ , \tag{38}$$

where we have written $(\psi^*)_{\dot{\beta}} = (\psi_\beta)^*$. Similarly, given a $(0,1/2)$ spinor $\bar{\chi}^{\dot{\alpha}}$, we can form a $(1/2,0)$ spinor through

$$\chi_\alpha := \epsilon_{\alpha\beta}(\bar{\chi}^*)^\beta\ . \tag{39}$$

Thanks to these handy relations, we only really ever need to deal with $(1/2,0)$ spinors.

For our next trick, let us try to connect the $(1/2,1/2)$ rep with the four-vector rep. To do so, let us define the set of four 2×2 matrices σ^μ ($\mu = 0,1,2,3$) by

$$\sigma^\mu_{\alpha\dot{\alpha}} = (\mathbb{I},\vec{\sigma})_{\alpha\dot{\alpha}}\ . \tag{40}$$

With this, we can form the object

$$\psi\sigma^\mu\bar{\chi} := \psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}\ . \tag{41}$$

Under Lorentz, we have

$$\begin{aligned}
\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} &\rightarrow \epsilon^{\alpha\lambda}(M_\lambda^{\ \beta}\psi_\beta)\sigma^\mu_{\alpha\dot{\alpha}}(\overline{M}^{\dot{\alpha}}_{\ \dot{\beta}}\,\bar{\chi}^{\dot{\beta}}) && (42)\\
&= -\psi_\beta[M^t\epsilon\sigma^\mu\overline{M}]^\beta_{\ \dot{\beta}}\bar{\chi}^{\dot{\beta}} && (43)
\end{aligned}$$

In the homework you will show that

$$[M^t\epsilon\sigma^\mu\overline{M}]^\beta_{\ \dot{\beta}} = \epsilon^{\beta\lambda}\Lambda^\mu_{\ \nu}\sigma^\nu_{\lambda\dot{\beta}}\ , \tag{44}$$

where Λ is the corresponding four-vector transformation. It follows that under Lorentz

$$\psi\sigma^\mu\bar{\chi} \rightarrow \Lambda^\mu_{\ \nu}\,\psi\sigma^\nu\bar{\chi}\ , \tag{45}$$

which transforms like a four vector. This justifies our notation for the upper index on σ^μ, and it also shows how the four-vector rep emerges from the $(1/2,1/2)$ rep.

Let us also define

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (\mathbb{I},-\vec{\sigma})^{\dot{\alpha}\alpha}\ . \tag{46}$$

Following the same steps as before, one can show that for any $(1/2,0)$ and $(0,1/2)$ spinors ψ and $\bar{\chi}$,

$$\begin{aligned} \bar{\chi}\bar{\sigma}^{\nu}\psi &:= \bar{\chi}_{\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\psi_{\alpha} && (47) \\ &\to \Lambda^{\mu}{}_{\nu}\,\bar{\chi}\bar{\sigma}^{\nu}\psi\ . && (48) \end{aligned}$$

Numerically, one also has

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{\mu}_{\dot{\beta}\beta}\ . \tag{49}$$

This relation means that the spinor indices on $\bar{\sigma}^{\mu}$ are consistent with raising and lowering with our good friend ϵ.

The σ^{μ} matrices also satisfy two very useful relations. The first follows from the tracelessness of the Pauli matrices, and reads

$$tr(\bar{\sigma}^{\mu}\sigma^{\nu}) = (\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}} = 2\eta^{\mu\nu}\ . \tag{50}$$

The second relation is

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\ (\sigma_{\mu})_{\beta\dot{\beta}} = -2\epsilon_{\alpha\beta}\,\epsilon_{\dot{\alpha}\dot{\beta}}\ . \tag{51}$$

1.4 Spinor Lagrangians

With all that spinor technology out of the way, we are now able to put together Lorentz-invariant Lagrangians for spinor fields. In doing so, however, there are two additional conditions. First, to describe a physical system, the action must be real. Since a spinor ψ is an inherently complex object, we must therefore have $\bar{\psi} = \epsilon\psi^{*}$ in our theory as well. Second, when we quantize later on we will find that spinors describe fermions. It turns out that for the quantum theory to connect in a reasonable way to a classical theory, the spinors must anticommute with each other (even in the classical theory). In particular,

$$\psi_{\alpha}\chi_{\beta} = -\chi_{\beta}\psi_{\alpha}, \qquad \psi_{\alpha}\bar{\chi}_{\dot{\beta}} = -\bar{\chi}_{\dot{\beta}}\psi_{\alpha}\ . \tag{52}$$

In fancy math language, spinors are said to be *Grassmann variables.*

For future use, it will be useful to have an explicit convention for the complex conjugation of multiple classical fields, whether they be bosonic (and commuting) or fermionic (and anticommuting). For either field type, we define for a single field

$$A^{\dagger} := A^{*}\ , \tag{53}$$

as well as

$$(A_1A_2\ldots A_n)^{*} := (A_1A_2\ldots A_n)^{\dagger} := A_n^{\dagger}\ldots A_2^{\dagger}A_1^{\dagger}\ . \tag{54}$$

Note that we have reversed the order with no additional signs, even for the fermion case. This convention is useful because it will match smoothly with the operation of Hermitian conjugation in the quantum theory, where we promote the fields to operators on a Hilbert space. Note as well that

$$(\chi\xi)^\dagger = (\epsilon^{\alpha\beta}\chi_\beta\xi_\alpha)^\dagger = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} = +\bar{\xi}\bar{\chi} \ . \tag{55}$$

At the very least, a sensible physical theory requires a kinetic term involving some spacetime derivatives. It turns out that the right form for a spinor is

$$\begin{aligned} \mathscr{L} &\supset \frac{1}{2}\psi i\sigma^\mu\partial_\mu\bar{\psi} + \frac{1}{2}\bar{\psi}i\bar{\sigma}^\mu\partial_\mu\psi & (56)\\ &= \bar{\psi}i\bar{\sigma}^\mu\partial_\mu\psi = \psi i\sigma^\mu\partial_\mu\bar{\psi} & (57)\end{aligned}$$

where you will verify the reality of the first line and the equalities in the second line in the homework.

We can also add a bilinear mass term for the spinor. If we only have a single spinor ψ (and its conjugate $\bar{\psi}$), the only Lorentz-invariant possibility is

$$\mathscr{L} \supset -\frac{1}{2}m\,\psi\psi - \frac{1}{2}m^*\bar{\psi}\bar{\psi} \ . \tag{58}$$

You might worry that these terms both vanish since ψ is anticommuting, but they do not. Note that

$$\chi\psi = \chi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\chi_\beta\psi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = \epsilon^{\beta\alpha}\psi_\alpha\chi_\beta = \psi^\beta\chi_\beta = \psi\chi \ , \tag{59}$$

where we see that the anticommutation of the spinors is cancelled by the antisymmetry of ϵ.

2 Dirac Fermions

Having spent all that time on two-component $(1/2,0)$ and $(0,1/2)$ spinors, we turn next to study four-component objects in the $(1/2,0)\oplus(0,1/2)$ representation. While the rep is reducible under Lorentz, it is irreducible if we also impose parity which exchanges j_A and j_B in (j_A, j_B). Parity turns out to be a good symmetry of electromagnetism, and therefore we would like to build it into our fields. This is why we'll use four-component Dirac fermions to describe electrons in QED.

2.1 Dirac Spinors

Consider a theory containing two $(1/2,0)$ spinors ξ and χ together with their conjugates. We will assume the theory has a global symmetry under the phase transformations

$$\xi(x) \to e^{i\varphi}\psi(x), \qquad \chi(x) \to e^{-i\varphi}\chi(x) \ . \tag{60}$$

The most general Lagrangian for this theory at bilinear order is

$$\mathscr{L} = \bar{\xi}i\bar{\sigma}^\mu\partial_\mu\xi + \bar{\chi}i\bar{\sigma}^\mu\partial_\mu\chi - m(\xi\chi + \bar{\xi}\bar{\chi}) \ . \tag{61}$$

The global symmetry allows the mixed $\chi\xi$ quadratic term, but it forbids $\chi\chi$ and $\xi\xi$.

When we quantize the theory, we will interpret m as the mass of a particle. However, since two fields are involved, it is not obvious how to relate the mass term to a specific particle. Using a four-component spinor containing both two-component spinors allows us to dodge this issue for the time being. We define the four-component *Dirac spinor* Ψ by

$$\Psi = \begin{pmatrix} \xi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} . \tag{62}$$

The conjugate of Ψ is thus

$$\Psi^\dagger = \left(\bar{\xi}_{\dot{\alpha}},\, \chi^\alpha\right) . \tag{63}$$

To go along with Ψ, we also generalize the σ^μ matrices to the 4×4 Dirac matrices γ^μ,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \tag{64}$$

where each of the matrix elements is itself a 2×2 matrix. Finally, let us define the barred conjugate $\bar{\Psi}$ to be

$$\bar{\Psi} = \Psi^\dagger \gamma^0 = \left(\chi^\alpha,\, \bar{\xi}_{\dot{\alpha}}\right) . \tag{65}$$

Note that here the bar denotes a conjugation operation, and is not part of the name of the Dirac spinor.

With these definitions in hand, we can rewrite the Lagrangian of Eq. (61) in a more compact form using Ψ. The result is

$$\mathscr{L} = \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi . \tag{66}$$

The mass term looks much nicer now.

2.2 Fun with γ Matrices

The γ matrices satisfy all kinds of useful relations that we will make use of when we compute scattering amplitudes in QED. The most important one is

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} . \tag{67}$$

It is easy to verify this relation using the definition of Eq. (64) and the properties of σ matrices. More generally, the γ matrices can be defined in any number of spacetime dimensions to be the minimal solutions of Eq. (67). In four dimensions, all the solutions can be shown to be equivalent to Eq. (64) up to a unitary change of basis for the four-component space.

Many of the other useful relations involve traces of γ matrices. Recall that the trace of any matrix M is defined to be

$$tr(M) = \sum_i M_{ii} = M_{11} + M_{22} + \ldots \tag{68}$$

Traces are linear,

$$tr(aM) = a\, tr(M) \ , \tag{69}$$

and they are cyclic,

$$tr(AB) = tr(BA), \quad tr(ABC) = tr(BCA) = tr(CAB), \quad etc. \tag{70}$$

From the explicit form of Eq. (64), we see that

$$tr(\gamma^\mu) = 0 \ . \tag{71}$$

This can be generalized, and it turns out that the trace of any odd number of γ^μ matrices vanishes. On the other hand, for even numbers we have

$$tr(\gamma^\mu\gamma^\nu) \ = \ 4\eta^{\mu\nu} \tag{72}$$

$$tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \ = \ 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \ , \tag{73}$$

which can both be derived using the cyclicity of the trace. An additional useful relation is

$$p_\mu p_\nu \gamma^\mu\gamma^\nu = \frac{1}{2} p_\mu p_\nu (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = 2\eta^{\mu\nu} p_\mu p_\nu = p^2 \ . \tag{74}$$

Finally, we have

$$\gamma^\mu\gamma_\mu = \eta_{\mu\nu}\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\eta_{\mu\nu} = 4 \ , \tag{75}$$

at least in four spacetime dimensions.

Let us also define the matrix $\gamma^5 = \gamma_5$ to be

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{76}$$

This matrix anticommutes with all the γ^μ,

$$\{\gamma^5, \gamma^\mu\} = 0 \ , \tag{77}$$

and satisfies

$$(\gamma^5)^2 = 1 \ . \tag{78}$$

Using the basis of Eq. (64), the explicit form of γ^5 is

$$\gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \ . \tag{79}$$

It is also conventional to define the projectors P_L and P_R to be

$$P_{L,R} = \frac{1}{2}(1 \mp \gamma^5) \ . \tag{80}$$

Applying this to a Dirac spinor, we get

$$P_L\Psi = \Psi_L = \left(\begin{array}{c} \xi_\alpha \\ 0 \end{array} \right), \qquad\qquad P_R\Psi = \Psi_R = \left(\begin{array}{c} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{array} \right) \ . \tag{81}$$

Note that

$$\Psi = \Psi_L + \Psi_R \ , \tag{82}$$

and

$$\bar{\Psi} i\gamma^\mu \partial_\mu \Psi = \bar{\Psi}_L i\gamma^\mu \partial_\mu \Psi_L + \bar{\Psi}_R i\gamma^\mu \partial_\mu \Psi_R \ , \tag{83}$$

as well as

$$\bar{\Psi}\Psi = \bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \ . \tag{84}$$

Neither result should be surprising given what you know about two-component spinors.

We will also sometimes evaluate traces involving γ^5. Clearly, we have

$$tr(\gamma^5) = 0 \ . \tag{85}$$

Since the trace of an odd number of γ matrices vanishes, we also have

$$tr(\gamma^5 \times (\text{odd number of } \gamma^\mu\text{s})) = 0 \ . \tag{86}$$

Cyclicity also implies

$$tr(\gamma^5 \gamma^\mu \gamma^\nu) = 0 \ . \tag{87}$$

However, the trace with four γ matrices is non-zero,

$$tr(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma} \ , \tag{88}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is completely antisymmetric with $\epsilon^{0123} = +1$.

References

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