PHYS 526 Notes #3: Interacting Scalars

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We have found that quantizing a scalar theory with $V(\phi) = \frac{1}{2}m^2\phi^2 + \Lambda$ produces a theory of identical free particles of mass m, in which the individual particles do not interact with each other at all. Due to the simplicity of this theory, we were able to find all the energy eigenstates and eigenvalues. On the other hand, this simplicity also means that the theory is not very interesting.

In this note we will look at a more complicated scalar theory with higher-order terms in the potential. Specifically, we will investigate the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \Lambda - \Delta V(\phi) \ . \tag{1}$$

In many cases, we will take

$$\Delta V = \frac{\lambda}{4!}\,\phi^4 \ , \tag{2}$$

where λ is a constant parameter. For this choice of ΔV, the classical equations of motion are

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\,\phi^3 \ . \tag{3}$$

This is a non-linear partial differential equation, implying that a linear combination of two solutions need no longer be a solution. In general, we don't know how to solve this equation analytically.

Turning to the Hamiltonian formulation of the classical theory, we find the conjugate momentum

$$\Pi(x) = \partial_t\phi(x) \ , \tag{4}$$

and the Hamiltonian

$$\begin{aligned} H &= \int d^3x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \Lambda\right] + \int d^3x\ \Delta V(\phi) & (5)\\ &:= \tilde{H}_0 + \Delta H \ , & (6) \end{aligned}$$

where $\Delta H = \int d^3x\,\Delta V(\phi)$.

1 Quantizing

To quantize this theory, we proceed just like before by elevating $\phi(x)$ and $\Pi(x)$ to operators on a Hilbert space and imposing canonical commutation relations on them (at equal times). Also as before, we can define $a(\vec{k})$ and $a^\dagger(\vec{k})$ operators at the fixed reference time $t_0 = 0$ according to

$$\begin{aligned} a(\vec{k}) &= i\int d^3\, e^{ik\cdot x}(\Pi - ik^0\phi)|_{t=0} && (7)\\ a^\dagger(\vec{k}) &= -i\int d^3\, e^{-ik\cdot x}(\Pi + ik^0\phi)|_{t=0} && (8)\\ & && (9) \end{aligned}$$

Using the commutators of ϕ and Π at $t = 0$, the commutators of these operators are

$$[a(\vec{k}), a^\dagger(\vec{p})] = (2\pi)^3 2k^0 \delta^{(3)}(\vec{k} - \vec{p}), \quad [a(\vec{k}), a(\vec{p})] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{p})]\ . \tag{10}$$

Building up the Hamiltonian at $t = 0$, we find

$$\begin{aligned} H &= \int \widetilde{dk}\, k^0 a^\dagger(\vec{k})a(\vec{k}) + \int d^3x\, \Delta V(\phi) && (11)\\ &= H_0 + \Delta H(t=0)\ . && (12) \end{aligned}$$

Note that even though we have built up the Hamiltonian from quantities defined at $t = 0$, it is time-independent (one of the assumptions of our quantum theory) and this expression holds for any time: $H(t) = H(0)$. We can also expand $\Delta H(0)$ in terms of the ladder operators, but the resulting expression will typically be complicated. For $\Delta V = \lambda\phi^4/4!$, it will involve products of four a and $a^\dagger$ operators.

As usual, the time dependence of operators (in the Heisenberg picture) is controlled by the Hamiltonian. In general, we have

$$\mathcal{O}(t) = e^{iHt0}\mathcal{O}(0)e^{-iHt} \tag{13}$$

Applying this to $a(\vec{k})$, we can form a time-dependent version:

$$a(t, \vec{k}) := e^{iHt}a(\vec{k})e^{-iHt}\ , \tag{14}$$

and similarly for $a^\dagger$. Applying this to the free theory, we find

$$a(t, \vec{k}) = e^{-ik^0t}a(\vec{k}) \qquad \text{(free theory)}\ . \tag{15}$$

Unfortunately, this simple result no longer holds in the interacting theory. In general, $a(t, \vec{k})$ will be a linear combination of products of multiple $a(\vec{k})$ and $a^\dagger(\vec{k})$ operators. Equivalently, in the Schrödinger picture the state $|p(t = 0)\rangle := a^\dagger(\vec{p})|0\rangle$ will evolve in time into other multiparticle states.

While we have managed to quantize this more complicated field theory, we have not been able to compute everything we would like. The major challenge is that, in general, we do not know how to calculate the time evolution of the states built from the a and $a^\dagger$ operators. Equivalently, we have not been able to find the eigenvalues and eigenvectors of the Hamiltonian. Whatever shall we do?

2 Perturbing Around the Free Theory

To proceed, we shall start with the free theory and expand around it. This expansion will be useful if ΔH is in some sense small relative to H_0. We will come back later and quantify this condition. The form of the expansion we will also be geared towards scattering, where particles come in from spatial infinity at $t \to -\infty$, scatter with each other, and travel off to spatial infinity $t \to +\infty$. Scattering is relatively easy to handle because particles that are separated by large distances are expected to behave like free particles (as long as the interaction is not too strong).

The first step in perturbation theory is to construct a set of eigenstates of H_0. Note that even thought $H_0 \neq H$, it is still a Hermitian operator and its eigenstates will therefore be a complete and orthonormalizable set. We already know what these are: the lowest state $|0\rangle$ and all the states that can be built by applying factors of $a^\dagger(\vec{p})$ to it. Note that we defined a and $a^\dagger$ in terms of $\phi(0,\vec{x})$ and $\Pi(0,\vec{x})$ at the reference time $t=0$.

In expanding around these H_0 eigenstates, we will make two physically reasonable assumptions. As long as the interaction is sufficiently small, they turn out to be justified.[1] They are:

1. There is a unique ground state $|\Omega\rangle$ of the full Hamiltonian with zero energy and momentum and non-zero overlap with $|0\rangle$:
$$\langle 0|\Omega\rangle \neq 0 \ . \tag{16}$$
Note that $|0\rangle$ can be different from $|\Omega\rangle$.

2. The next state in the spectrum is an isolated one-particle state with momentum $\vec{p}$ and energy $E = \sqrt{\vec{p}^2 + M^2}$ for some mass M (possibly different from m). By isolated, we mean that there is a non-zero energy gap above $|\Omega\rangle$, and another energy gap between the one-particle state (in the Lorentz frame with $\vec{p} = \vec{0}$) and the next set of states.

The first assumption means that the interaction does not change the vacuum of the theory in to radical a way. The second implies that we will still be able to associate the field $\phi(x)$ with a specific particle species.

2.1 The Interaction Picture

To expand the full theory around the free theory, let us begin by writing $\phi(0,\vec{x})$ in terms of the ladder operators by inverting Eqs. (7,8):
$$\phi(0,\vec{x}) = \int \widetilde{dk}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\right] \ . \tag{17}$$
To evolve the field to later times, we use the full Hamiltonian:
$$\phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt} \ . \tag{18}$$

[1]However, they can be explicitly violated at strong coupling.

We don't know what this is because we don't know how to move H through the $a(\vec{k})$ and $a^\dagger(\vec{k})$ operators that make up $\phi(0,\vec{x})$.

In the face of this challenge let's do something easier and define the time-evolved field in the *interaction picture* by

$$\begin{aligned} \phi_I(t,\vec{x}) &= e^{iH_0 t}\phi(0,\vec{x})e^{-iH_0 t} && (19)\\ &= \int \widetilde{dk}\, \left[a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right] , && (20) \end{aligned}$$

where H_0 is the free Hamiltonian at $t=0$. Here, the time evolution is simple because we do know how to commute H_0 with a and $a^\dagger$. We can also generalize this definition to any other local operator: $\mathcal{O}_I(t) = e^{iH_0 t}\mathcal{O}(0)e^{-iH_0 t}$.

Before moving on, let us briefly take note of the time dependence of the full Hamiltonian and its pieces. We have

$$\begin{aligned} H(t) &= e^{iHt}H(0)e^{-iHt} \;=\; H(0) && (21)\\ &= e^{iHt}H_0 e^{-iHt} + e^{iHt}\Delta H(0)e^{-iHt} && (22)\\ &:= \tilde{H}_0(t) + \Delta H(t) . && (23) \end{aligned}$$

While the full Hamiltonian is time-independent (since it commutes with itself), the terms within it need not be. We have defined here a time-dependent version $\tilde{H}_0(t)$ of the free Hamiltonian such that $H_0 = \tilde{H}_0(0)$. The tilde is to distinguish it from H_0, which is given by the explicit expression in Eq. (11).

The interaction picture is useful because it factors out the time evolution due to the free Hamiltonian. We can relate any Heisenberg-picture operator to the interaction picture version by

$$\mathcal{O}(t) = U^\dagger(t)\mathcal{O}_I(t)U(t) , \tag{24}$$

where

$$U(t) = e^{iH_0 t}e^{-iHt} . \tag{25}$$

Differentiating, this implies

$$i\partial_t U(t) = \Delta H_I(t)U(t) , \tag{26}$$

where $\Delta H_I(t) = e^{iH_0 t}\Delta H(0)e^{-iH_0 t}$ is the interaction Hamiltonian in the interaction picture. Note that if $\Delta H = \int d^3x\, \lambda^{(m,n)}\Pi^m\phi^n$, we have $\Delta H_I = \int d^3x\, \lambda^{(m,n)}\Pi_I^m\phi_I^n$.

The solution to Eq. (26) for $t>0$ is derived in detail in Peskin & Schroeder [1], and is given by *Dyson's equation*:

$$\begin{aligned} U(t) &= T\left\{\exp\left[-i\int_0^t dt'\Delta H_I(t')\right]\right\} && (27)\\ &= \mathbb{I} + (-i)\int_0^t dt_1\, \Delta H_I(t_1) + \frac{(-i)^2}{2!}\int_0^t dt_1 \int_0^t dt_2\, T\{\Delta H_I(t_1)\, \Delta H_I(t_2)\} + \ldots && (28) \end{aligned}$$

where the first line is just a compact shorthand for the second. This expression certainly has the right boundary condition, and it is straightforward to check that it satisfies Eq. (26) by explicit differentiation. For $t < 0$, the solution is

$$U(t) = T' \left\{ \exp\left[-i \int_0^t dt' \Delta H_I(t') \right] \right\} \ , \tag{29}$$

where T' denotes reverse time ordering

$$T'\{\phi(x_1)\phi(x_2)\} = \Theta(t_1 - t_2)\, \phi(x_2)\phi(x_1) + \Theta(t_2 - t_1)\, \phi(x_1)\phi(x_2) \ . \tag{30}$$

To combine both cases into a simple notation, let us define $\widetilde{T}$ to be time ordering for $t > 0$ and reverse time ordering for $t < 0$. Thus,

$$U(t) = \widetilde{T} \left\{ \exp\left[-i \int_0^t dt' \Delta H_I(t') \right] \right\} \ . \tag{31}$$

In terms of $U(t)$, we now have an analytic (but very complicated) relation between $\phi(t, \vec{x})$ and $\phi_I(t, \vec{x})$.

We can generalize $U(t)$ by defining

$$U(t_2, t_1) = U(t_2)U^\dagger(t_1) = e^{iH_0 t_2} e^{-iH(t_2 - t_2)} e^{-iH_0 t_1} \ . \tag{32}$$

It can be shown that this quantity is equal to

$$U(t_2, t_1) = \widetilde{T} \left\{ \exp\left[-i \int_{t_1}^{t_2} dt' \Delta H_I(t') \right] \right\} \ , \tag{33}$$

where $\widetilde{T}$ is time ordering for $t_2 > t_1$ and reverse time-ordering for $t_2 < t_1$.

The $U(t_1, t_2)$ operator has some very nice properties. They include:

1. $U(t_1, t_1) = \mathbb{I}$
2. $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$
3. $U^\dagger(t_2, t_1) = U^{-1}(t_2, t_1) = U(t_1, t_2)$

By definition, $U(t, 0) = U(t)$. This implies trivially that $\phi(x) = U^\dagger(t, 0)\phi_I(x)U(t, 0)$. Thus, for any product of fields (where $\phi_i = \phi(x_i)$):

$$\begin{aligned} \phi_n \phi_{n-1} \ldots \phi_1 &= U^\dagger(t_n, 0)\phi_{I_n} U(t_n, 0) U^\dagger(t_{n-1}, 0)\phi_{I_{n-1}} \ldots U(t_2, 0)U^\dagger(t_1, 0)\phi_{I_1} U(t_1, 0) \\ &= U^\dagger(t_n, 0)\, \phi_{I_n}\, U(t_n, t_{n-1})\, \phi_{I_{n-1}} \ldots \phi_{I_2}\, U(t_2, t_1)\, \phi_{I_1}\, U(t_1, 0) \ . \end{aligned} \tag{34}$$

In this expression, you can think of the U factors as transfer operators that connect the interaction-picture operators at different times.

2.2 To Infinity and Beyond

When we specialize to scattering, we will be interested in the limits of $t \to \pm\infty$. This is useful because going to temporal infinity allows us to isolate the vacuum. Let us choose the constant in the free Lagrangian such that $H|\Omega\rangle = 0$ and $H_0|0\rangle = E_0|0\rangle$. If we apply the $U(t, 0)$ operator to the true vacuum $|\Omega\rangle$, we find

$$\begin{aligned} U(-\tau, 0)|\Omega\rangle &= e^{-iH_0\tau}e^{iH\tau}|\Omega\rangle && (35) \\ &= e^{-iH_0\tau}\left(\sum_{n=0}^{\infty}|n\rangle\langle n|\right)|\Omega\rangle && (36) \\ &= e^{-iE_0\tau}|0\rangle\langle 0|\Omega\rangle + \sum_{n=1}^{\infty}e^{-iE_n\tau}|n\rangle\langle n|\Omega\rangle\ . && (37) \end{aligned}$$

where we have inserted unity in the form of a complete set of H_0 eigenstates in the second line. This result simplifies if we multiply by $e^{iE_0\tau}$ and take the limit $\tau \to \infty(1 - i\epsilon)$ with $0 < \epsilon \ll 1$. Specifically,

$$\lim_{\tau\to\infty(1-i\epsilon)} e^{iE_0\tau}U(-\tau, 0)|\Omega\rangle = |0\rangle\langle 0|\Omega\rangle\ . \quad (38)$$

All the other terms vanish exponentially quickly in this limit.

Consider now the the vacuum expectation value of the product of n fields. Noting that $U(t, 0) = U(t, \tau)U(\tau, 0)$ and $U^\dagger(t, 0) = U^\dagger(\tau, 0)U^\dagger(\tau, t)$, and making use of the result of Eq. (34), we can write

$$\begin{aligned} \langle\Omega|\phi_n \ldots \phi_1|\Omega\rangle &= && (39) \\ &\langle\Omega|U^\dagger(\tau, 0)U(\tau, t_n)\phi_{I_n}U(t_n, t_{n-1}) \ldots U(t_2, t_1)\phi_{I_1}U(t_1, \tau)U(-\tau, 0)|\Omega\rangle\ . \end{aligned}$$

Taking the limit $\tau \to \infty(1 - i\epsilon)$ then gives

$$\begin{aligned} \langle\Omega|\phi_n \ldots \phi_1|\Omega\rangle &= && (40) \\ &\lim_{\tau}|\langle 0|\Omega\rangle|^2\, \langle 0|U(\tau, t_n)\phi_{I_n}U(t_n, t_{n-1}) \ldots U(t_2, t_1)\phi_{I_1}U_(t_1, -\tau)|0\rangle\ . \end{aligned}$$

This is still very messy, but it can be simplified in the case of time-ordered products.

First, let us evaluate $|\langle 0|\Omega\rangle|^2$. Taking the inner product Eq. (37) with itself and setting $\tau \to \infty'$, one can show that

$$|\langle 0|\Omega\rangle|^2 = 1 \Big/ \lim_{\tau\to\infty(1-i\epsilon)} T\left\{\exp\left[-i\int_{-\tau}^{\tau} dt'\Delta H_I(t')\right]\right\} \quad (41)$$

This is nice and compact.

Turning next to the time-ordered product of n fields, the expectation value in the vacuum reduces to

$$\langle\Omega|T\{\phi_n \ldots \phi_1\}|\Omega\rangle = \lim_{\tau\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_{I_1} \ldots \phi_{I_n} \exp\left[-i\int_{-\tau}^{\tau} dt'\Delta H_I(t')\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-\tau}^{\tau} dt'\Delta H_I(t')\right]\right\}|0\rangle}\ . \quad (42)$$

The reason for this much simpler form is that the time-ordering means that everything now commutes inside it, allowing us to collect and contract all the $U(t_i, t_j)$ factors. This will turn out to be a very useful result.

We will see shortly that time-ordered products of fields are the main quantities of interest for scattering. Since they come up so much, they are given a special name and an abbreviated notation. We call the time-ordered product of n fields the n-point function $G^{(n)}(x_1, \ldots, x_n)$ and we will often write it as

$$\langle\Omega|T\{\phi_1 \ldots \phi_n\}|\Omega\rangle = G^{(n)}(x_1, \ldots, x_n) = \langle\phi_1 \ldots \phi_n\rangle \ . \tag{43}$$

Yay.

2.3 Computing Stuff, Finally

After all this formalism, let's actually compute something explicitly. The way Eq. (42) is used in practice is as an expansion in powers of ΔH_I. In this form, we can evaluate the expectation values of the interaction-picture fields by expanding them in terms of the a and $a^\dagger$ ladder operators as in Eq. (20). Since we know how these act on $|0\rangle$ and how they commute with each other, we have all the tools we need to do so.

To be concrete, let us specialize to the case of

$$\Delta H = \int d^3x \, \frac{g}{3!}\phi^3 \ . \tag{44}$$

It follows that $\Delta H_I(t) = \int d^3x \frac{g}{3!}\phi_I^3(t, \vec{x})$. The easiest thing to compute is the expectation value of the identity operator. For this, Eq. (42) simply gives

$$\langle\mathbb{I}\rangle = 1 \ . \tag{45}$$

Not so bad at all.

A slightly more challenging quantity is the 2-point function,

$$\langle\phi_1\phi_2\rangle = \frac{\langle 0|T\left\{\phi_{I_1}\phi_{I_2}\exp\left[-i\int d^4z \, \frac{g}{3!}\phi_I^3(z)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int d^4z \, \frac{g}{3!}\phi_I^3(z)\right]\right\}|0\rangle} \ . \tag{46}$$

To leading non-trivial order, which is g^0 in this case, the numerator is

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\}|0\rangle = D_F(x_1 - x_2) \ , \tag{47}$$

while the denominator is equal to unity. There are higher-order contributions, but it is reassuring that we reproduce the free theory result at lowest order.

A quantity that only arises at order g^1 in this theory is the 3-point function. Evaluating it at this order,

$$G^{(3)}(x_1, x_2, x_3) = \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\left[1 - i\int d^4z \frac{g}{3!}\phi_I^3(z)\right]|0\rangle + \ldots \tag{48}$$

$$= 0 + \frac{(-i)g}{3!}\int d^4z \ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I^3(z)\}|0\rangle + \ldots \tag{49}$$

The terms omitted in each line are higher-order in g (g^2 or higher). In particular, at this order we can set the denominator to unity. The second line can be evaluated by expanding the fields in terms of ladder operators, arranging their time ordering, and commuting the ladder operators to put everything in normal order. This requires a lot of brute-force calculation to do in general, but fortunately, there is a systematic way of computing them called Wick's Theorem.

2.4 Wick's Theorem

To compute the vacuum matrix elements of time-ordered products of interacting fields, Eq. (42) tells us we need to evaluate a lot of time-ordered products of interaction-picture fields sandwiched between $\langle 0|$ and $|0\rangle$. Working out the operator expectation values is straightforward, but it can become very tedious due to the many terms that arise along the way. Fortunately, there is a straightforward way to evaluate them based on a result called *Wick's Theorem*.

Wick's Theory is a statement about free fields that also applies to fields in the interaction picture. In words, it is

$$T\left\{\phi(x_1)\ldots\phi(x_n)\right\} = N\left\{\phi(x_1)\ldots\phi(x_n) + \text{all contractions}\right\} \ . \tag{50}$$

We will explain what we mean by a contraction below, but for now let us just say that it reduces the number of field operators by two units. The utility of this formula is that a normal-ordered operator vanishes when sandwiched between $\langle 0|$ and $|0\rangle$ unless it is proportional to the identity. This means that only the terms on the right-hand side that are fully contracted, such that no field operators are left, can contribute to the vacuum expectation value.

To describe Wick's theorem, it will be useful to split up our free-field operator into two pieces:

$$\phi(x) = \int \widetilde{dk}\ \left[a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right] = \phi_-(x) + \phi_+(x) \ , \tag{51}$$

with[2]

$$\phi_-(x) = \int \widetilde{dk}\ a(\vec{k})e^{-ik\cdot x}, \qquad \phi_+(x) = \int \widetilde{dk}\ a^\dagger(\vec{k})e^{ik\cdot x} \ . \tag{52}$$

From this definition, we find that

$$[\phi_-(x_1), \phi_-(x_2)] = 0 = [\phi_+(x_1), \phi_+(x_2)] \ , \tag{53}$$

together with

$$\phi_-^\dagger(x) = \phi_+(x) \ , \tag{54}$$

[2]Note that my + and − are flipped relative to P&S.

as well as

$$[\phi_-(x_1), \phi_+(x_2)] = \int \widetilde{dk}\, e^{-ik\cdot(x_1-x_2)} := D(x_1 - x_2) \ . \tag{55}$$

In terms of ϕ_- and ϕ_+, a product of fields will be normal-ordered if and only if all the ϕ_- pieces are written to the right of all the ϕ_+ pieces.

Since $T\{\mathbb{I}\} = N\{\mathbb{I}\}$ and $T\{\phi(x)\} = N\{\phi(x)\}$ are both trivial, let's look at the product of two fields:

$$\begin{aligned}
T\{\phi(x_1)\phi(x_2)\} &= \Theta(t_1-t_2)\,[\phi_-(x_1)+\phi_+(x_1)]\,[\phi_-(x_2)+\phi_+(x_2)] && (56)\\
&\quad +\Theta(t_2-t_1)\,[\phi_-(x_2)+\phi_+(x_2)]\,[\phi_-(x_1)+\phi_+(x_1)] \\
&= \Theta_{12}\,(\phi_{1-}\phi_{2-}+\phi_{1+}\phi_{2+}+\phi_{1+}\phi_{2-}+\phi_{2+}\phi_{1-}+[\phi_{1-},\phi_{2+}]) && (57)\\
&\quad +\ \Theta_{21}\,(\phi_{2-}\phi_{1-}+\phi_{2+}\phi_{1+}+\phi_{2+}\phi_{1-}+\phi_{1+}\phi_{2-}+[\phi_{2-},\phi_{1+}]) \\
&= \phi_{1-}\phi_{2-}+\phi_{1+}\phi_{2-}+\phi_{2+}\phi_{1-}+\phi_{1+}\phi_{2-} && (58)\\
&\quad +\ \Theta(t_1-t_2)D(x_1-x_2)+\Theta(t_2-t_1)D(x_2-x_1) \\
&= N\{\phi_1\phi_2\}+D_F(x_1-x_2)\ . && (59)
\end{aligned}$$

In the third line, we have used $\Theta_{12}+\Theta_{21}=1$, while in the fourth we have combined the two D functions into a D_F.[3] Let us now define the contraction of two fields to be

$$\contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2) = D_F(x_1-x_2)\ . \tag{60}$$

With this in place, we have

$$T\{\phi(x_1)\phi(x_2)\} = N\{\phi(x_1)\phi(x_2) + \contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2)\}\ . \tag{61}$$

Note that since D_F is just a function proportional to the unit operator, we can freely move it inside the normal ordering. This is consistent with our statement of Wick's Theorem.

Doing the same calculation for three fields, one obtains

$$\begin{aligned}
T\{\phi_1\phi_2\phi_3\} &= N\{\phi_1\phi_2\phi_3+\phi_1\,\contraction{}{\phi_2}{}{\phi_3}\phi_2\phi_3+\phi_2\,\contraction{}{\phi_3}{}{\phi_1}\phi_3\phi_1+\phi_3\,\contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2\} && (62)\\
&= N\{\phi_1\phi_2\phi_3\phi_4+\phi_1 D_F(x_2-x_3)+\phi_2\,D_F(x_1-x_3)+\phi_3\,D_F(x_1-x_2)\}. && (63)
\end{aligned}$$

For four fields,

$$\begin{aligned}
T\{\phi_1\phi_2\phi_3\phi_4\} = N\{&\phi_1\phi_2\phi_3\phi_4+\contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2\,\phi_3\phi_4+\contraction{}{\phi_1}{}{\phi_3}\phi_1\phi_3\,\phi_2\phi_4+\contraction{}{\phi_1}{}{\phi_4}\phi_1\phi_4\,\phi_2\phi_3 \\
&+\contraction{}{\phi_2}{}{\phi_3}\phi_2\phi_3\,\phi_1\phi_4+\contraction{}{\phi_2}{}{\phi_4}\phi_2\phi_4\,\phi_1\phi_3+\contraction{}{\phi_3}{}{\phi_4}\phi_3\phi_4\,\phi_2\phi_3 \qquad (64)\\
&+\contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2\,\contraction{}{\phi_3}{}{\phi_4}\phi_3\phi_4+\contraction{}{\phi_1}{}{\phi_3}\phi_1\phi_3\,\contraction{}{\phi_2}{}{\phi_4}\phi_2\phi_4+\contraction{}{\phi_1}{}{\phi_4}\phi_1\phi_4\,\contraction{}{\phi_2}{}{\phi_3}\phi_2\phi_3\}\ .
\end{aligned}$$

Oof.

[3] Check this for yourself by expanding out the fields in the definition of D_F.

The proof of Wicks' theorem goes by induction. Suppose it holds for any product of n fields. For $(n+1)$ fields labelled such that $t_1 > t_2 > \ldots$, we get

$$T\{\phi_1\phi_2\ldots\phi_{n+1}\} = \phi_1 T\{\phi_2\ldots\phi_{n+1}\} \tag{65}$$

$$= (\phi_{1+} + \phi_{1-})\, N\{\phi_2\ldots\phi_{n+1} + \text{contractions}'\} \tag{66}$$

where contractions′ means all contractions that do not involve ϕ_1. In this expression, the ϕ_{1+} term is already in normal order, so we only need to move ϕ_{1-} through to the right. This will eventually produce something that is normal-ordered together with a bunch of contractions, noting that $[\phi_-(x_1), \phi_+(x_i)] = D_F(x_1 - x_i)$ since we have $t_1 > t_i$ for all i. All that remains to show is that all possible contractions involving ϕ_1 are produced. We can do this by giving an algorithm to build a given contraction given ϕ_{1-} on the left and the terms already assumed to be present in the inductive step. I won't do this explicitly, but it isn't too difficult with a bit of fiddling.

Our statement and (partial) proof of Wick's Theorem was for free scalar fields. However, it also applies to scalar fields in the interaction picture since they obey the same commutation relations and have the same ladder operator expansions as free fields. Therefore Wick's theorem will allow us to systematically evaluate the operators that arise in the vacuum expectation values of time-ordered products of fields.

2.5 Feynman Diagrams and Rules

Let us return to the interacting scalar theory with $\Delta V = g\phi^3/3!$ and evaluate the leading-order expression for the 3-point function we found in Eqs. (48,49). Applying Wick's Theorem,

$$\begin{aligned}\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\,\phi_I^3(z)\}|0\rangle \;=\; & \\ (3!)D_F(x_1-z)D_F(x_2-z)D_F(x_3-z) + 3\,D_F(x_1-z)D_F(x_2-x_3)D_F(z-z) & \\ +\; 3\,D_F(x_2-z)D_F(x_1-x_3)D_F(z-z) \;+\; 3\,D_F(x_3-z)D_F(x_1-x_2)D_F(z-z) &\end{aligned} \tag{67}$$

Note that only fully contracted terms in the Wick expansion contribute to the matrix element. The numerical prefactors are just the number of ways to get the specific contraction.

We can represent this set of contractions in pictures. To each different term, we associate a *Feynman diagram*. The set of diagrams corresponding to the result of Eq. (68) is shown in Fig. 1. The rules for drawing a diagram are as follows:

1. For each coordinate x_i, draw a dot.
2. A contraction producing a factor of $D_F(x_i - x_j)$ is represented by a line connecting the dot for x_j to the dot for x_i.
3. Each factor of $D_F(z-z)$ is depicted as a loop connecting z to itself.

The utility of these rules is that we can invert them, using diagrams to figure out all the possible contractions, and then associating a number to each diagram. Instead of wading through a mire of raising and lowering operators, we only need to connect up some dots.

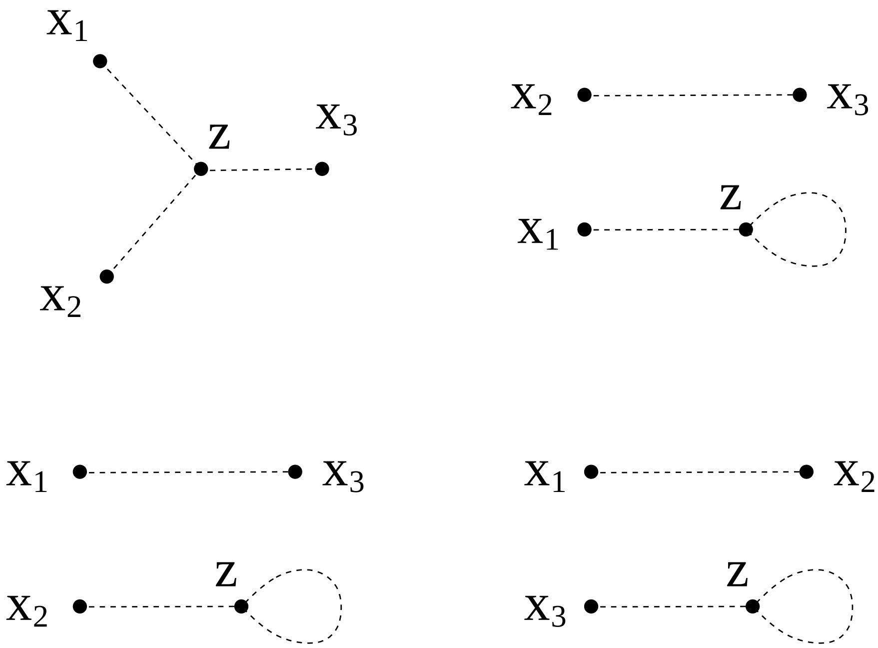

Figure 1: Feynman diagrams for the 3-point function at $\mathcal{O}(g)$ of Eq. (68).

Let us now formulate a set of *Feynman rules* for computing time-ordered vacuum matrix elements of fields in the $g\phi^3/3!$ theory. These rules will specify how to draw a set of Feynman diagrams for a given matrix element and assign a numerical value to each of them. They are based on the master formula of Eq. (42), which we almost always treat by expanding the exponentials it contains in a power series in g up to some fixed order g^N. At this order, the perturbation theory estimate for the matrix element is given by the sum of each of the individual g^0, $g^1, \ldots$, g^N contributions.

The Feynman rules to compute the g^M contribution to the n-point matrix element $G^{(n)}(x_1, \ldots, x_n) = \langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle$ are:

1. Draw a dot for each x_i coordinate. We call these external points.

2. Draw another M dots and associate a coordinate z_j $(j = 1, 2, \ldots M)$ to each of them. We call these vertices.

3. Draw Feynman diagrams by connecting the dots with lines in all possible ways following two simple rules:
   a) Each external point has a single line connected to it.
   b) Each vertex has three lines connected to it.

4. Remove all the diagrams in which there is a *vacuum bubble* – a vertex point that not connected to any of the external points by any path.

5. Write the number corresponding to each Feynman diagram:
   a) For every vertex with coordinate z_j, write a factor of $-ig\int d^4z_j$.

b) For every line connecting points a and b (external or vertex), add a factor of the Feynman propagator $D_F(a-b)$.
c) Any diagram with an unconnected dot or an unpaired line is equal to zero.
d) Multiply the result for each diagram by $1/M!$ and a symmetry factor.

The net g^M contribution is then the sum of all the diagrams. We will explain rules 4 and 5d) below.

To how this works in practice, let us begin with the 2-point function and work to second order (g^2) in the perturbative expansion. From our master formula, we have

$$G^{(2)}(x_1,x_2) = \langle 0|T\left\{\phi_1\phi_2\left[1+\frac{(-i)g}{3!}\int d^4z\,\phi_z^3+\frac{(-i)^2g^2}{3!3!2!}\int d^4z_1\int d^4z_2\,\phi_{z_1}^3\phi_{z_2}^3\right]\right\}|0\rangle \Big/ \langle 0|T\left\{1+\frac{(-i)g}{3!}\int d^4z\,\phi_z^3+\frac{(-i)^2g^2}{3!3!2!}\int d^4z_1\int d^4z_2\,\phi_{z_1}^3\phi_{z_2}^3\right\}|0\rangle \tag{68}$$

Following the rules and drawing the Feynman diagrams, we find the set shown in Fig. 2. At leading order g^0, we only have the first diagram in the figure and associate to it the number

$$G^{(2)}(x_1,x_2)|_{g^0} = D_F(x_1-x_2)\ . \tag{69}$$

This matches what we would have found from applying Wick's Theorem to the matrix element. There are no contributions proportional to g^1 because it is impossible to connect up all the dots and all the lines. Again, this matches what the matrix element would produce. Going to g^2, we find the second row of diagrams in Fig. 2. Their sum is

$$\begin{aligned} G^{(2)}(x_1,x_2)|_{g^2} = & \\ & \left[\frac{1}{2}\right]\left(\frac{2!}{2!}\right)(-ig)^2\int d^4z_1\int d^4z_2\,D_F(x_1-z_1)D_F(z_1-z_2)D_F(z_1-z_2)D_F(z_2-x_2) \\ & +\left[\frac{1}{2}\right]\left(\frac{2!}{2!}\right)(-ig)^2\int d^4z_1\int d^4z_2\,D_F(x_1-z_1)D_F(z_1-z_2)D_F(z_2-z_2)D_F(z_1-x_2) \\ & +\left[\frac{1}{4}\right]\left(\frac{2!}{2!}\right)(-ig)^2\int d^4z_1\,D_F(x_1-z_1)D_F(z_1-z_1)\int d^4z_2\,D_F(x_2-z_2)D_F(z_2-z_2) \end{aligned} \tag{70}$$

The factors of (2!/2!) come from the (1/2!) in the expansion of the exponential together with the fact that since z_1 and z_2 are integrated over, diagrams differing only by the permutation of z_1 and z_2 have the same value. The numerical prefactors in the square brackets $(1/2, 1/2, 1/4)$ are the symmetry factors. Note as well that we have omitted the lower row of diagram in Fig. 2 (surrounded by dotted boxes) because they have a vacuum bubble, in which one or more vertices are not connected to any of the external points in any way. The full 2-point function in the interacting theory up to order g^2 in perturbation theory is therefore

$$G^{(2)}(x_1,x_2) = G^{(2)}(x_1,x_2)|_{g^0} + G^{(2)}(x_1,x_2)|_{g^2} + \dots\ . \tag{71}$$

This is just a number to be evaluated.

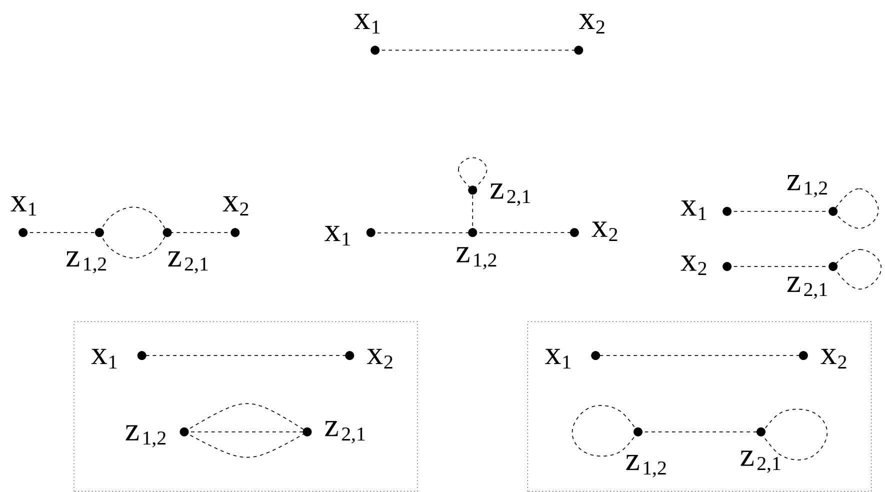

Figure 2: Feynman diagrams for the 2-point function at $\mathcal{O}(g^2)$.

Most of the Feynman rules are easy to relate to what's going on in matrix element after applying Wick's Theorem. However, rules 4 and 5d) probabaly seems a bit funny. For rule 4, the reason why we can drop these diagrams is that they are cancelled off by the expansion of the denominator in Eq. (42). We can see this explicitly here by studying the matrix element. Applying Wick's theorem to the denominator in Eq. (68), we get

$$\begin{aligned}\text{Denom.} &= 1 + \left[\frac{1}{12}\right](-i)^2 g^2 \int d^4z_1 \int d^4z_2 \; [D_F(z_1 - z_2)]^3 \\ &\quad + \left[\frac{1}{8}\right](-i)^2 g^2 \int d^4z_1 \int d^4z_2 \; D_F(z_1 - z_2) D_F(z_1 - z_1) D_F(z_2 - z_2) \\ &:= (1 + \Delta) \; .\end{aligned} \tag{72}$$

Using $1/(1+\Delta) = 1 - \Delta + \ldots$ the net result up to order g^2 is

$$G^{(2)}(x_1, x_2) = \left[D_F(x_1 - x_2) + g^2(\ldots)\right] \times (1 - \Delta) \tag{73}$$

$$= D_F(x_1 - x_2)(1 - \Delta) + g^2(\ldots) \; , \tag{74}$$

where the $g^2(\ldots)$ term corresponds to all the stuff in the numerator of the matrix element proportional to g^2. Some of the terms in the numerator will be the diagrams in the bottom row of Fig. 2 that contain vacuum bubbles. Using the Feynman rules, it is easy to show that they sum to to $D_F(x_1 - x_2)\Delta$. These diagrams are therefore cancelled off exactly by the expansion of the denominator. It turns out that such a cancellation between the denominator and diagrams in the numerator containing vacuum bubbles is a general feature, and a proof can be found in Peskin &Schroeder [1]. This explains the origin of rule 4.

Some diagrams also require a symmetry factor. In many cases, summing over all the contractions cancels off the (1/3!) in our definition of $\Delta V = g\phi^3/3!$. However, sometimes this cancellation is incomplete and we need to correct for it with a symmetry factor for the diagram. In general, it is equal to one divided by the number of ways to that internal lines can be reconnected to give the same diagram [1, 2]. As a practical matter, most people just work out the number of contractions corresponding to each diagram by counting contractions using Wick's theorem.

Finally, let us also mention that the denominator factor of 2! from the expansion of the exponential cancelled against the factor of 2! arising from the fact that for each diagram with fixed z_1 and z_2, there was a corresponding diagram with $z_1 \leftrightarrow z_2$ having the same numerical value. This is also a general feature at any order because all the internal z_i coordinates are integrated over. Thus, we can slightly modify our Feynman rule 5d) and leave out the $1/M!$ factor (at order g^M) with the understanding that we are only to include those diagrams that remain distinct when the z_i coordinates are permuted.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[2] M. Srednicki, "Quantum field theory," *Cambridge, UK: Univ. Pr. (2007) 641 p*