PHYS 526 Notes #10: Quantizing the Photon Field

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Our remaining task is to justify the photon propagator and external state sums we presented in the Feynman rules for QED. In doing so, we will also gain some intuition about how the gauge invariance of the theory manifests itself in terms of Feynman diagrams. It also turns out that the methods we used to develop the quantum theories of scalars and fermions are somewhat inconvenient for treating vector fields related to gauge invariance (such as the photon). For this, we will instead use the path integral formulation of quantum mechanics to handle the photon field.

1 Path Integral Quantization

You may or may not have encountered path integrals in your previous quantum mechanics courses. If not, don't worry too much since we will present everything you need to know here. Also, don't worry too much about mathematical rigour. We don't really know how to define path integrals in a fully rigourous way. Nevertheless, they are extremely useful and they give a very nice intuitive picture of quantum mechanics. My favourite treatment of path integrals is Appendix A of Ref. [1], and much of these notes follows this text.

1.1 Path Integrals for Bosonic Fields

We defined the functional derivative previously according to

$$\frac{\delta f(\phi(x))}{\delta\phi(x')} = \frac{\partial f}{\partial\phi}\,\delta^{(4)}(x-x')\ . \tag{1}$$

This is just a continuous generalization of $\partial_\mu f(x) = \delta_\mu^{\ \nu}\,(\partial f/\partial x^\nu)$. Functional (or path) integrals will turn out to be an analagous generalization.

Let us first define the functional integral of the field $\phi(t,\vec{x})$ at the fixed time t_i. For this, we divide space into a lattice of points $\{\vec{x}_j\}$ labelled by $j = 1,\ldots M$. The functional integral at time t_i is then defined to be

$$\int[\mathscr{D}'\phi_i] \sim \lim_{M\to\infty}\prod_{j=1}^{M}\left[\int_{-\infty}^{\infty}d\phi(t_i,\vec{x}_j)\right]\ , \tag{2}$$

where the squiggle means equality up to an overall factor. The functional integral is therefore just the product of integrals over the field value at each points in space. As you can imagine, taking the *continuum limit* $M\to\infty$ is highly non-trivial, and not even necessarily well-defined.

To define the full path integral, let us take the time interval $[t', t'']$ and subdivide it into $(N+1)$ pieces. Take $t' = t_0$ and $t'' = t_{N+1}$, with t_i as the i-th intermediate time slice. The full path integral with fixed endpoints $\phi(t', \vec{x}) = \phi'(\vec{x})$ and $\phi(t'', \vec{x}) = \phi''(\vec{x})$ is then

$$\int [\mathscr{D}\phi]_{\phi'}^{\phi''} \sim \lim_{N\to\infty} \prod_{i=1}^{N} \int [\mathscr{D}'\phi_i] \ . \tag{3}$$

Note that the endpoints are fixed, and do not get integrated over. In many cases we will take $t' \to -\infty$, $t'' \to \infty$, and force ϕ', $\phi'' \to 0$. We will write the resulting path integral in this case as simply $\int [\mathscr{D}\phi]$. It is equivalent to the more symmetric expression

$$\int [\mathscr{D}\phi] \sim \prod_x \left[\int_{-\infty}^{\infty} d\phi(t, \vec{x})\right] \ . \tag{4}$$

You should think of this as a sum over all possible spacetime configurations of the field ϕ. You should also think of the product over points in spacetime $x = (t, \vec{x})$ as the product over a lattice of points $x_i = (t_i, \vec{x}_i)$ in the limit that the lattice spacing is taken to zero.

The bad new about path integrals is that this definition is rather less than precise. However, the good news is that we only ever do a small number of different types of integrals. The first, and easiest, is

$$\int [\mathscr{D}\phi]\, \delta[\phi - \phi']\, S[\phi] = S[\phi'] \ , \tag{5}$$

where $S[\phi]$ is any functional of the fields and ϕ' is a specific function (*i.e.* a specific field configuration). The delta functional can also be written as a path integral, just like we have for the usual delta function. To derive it, note first that the delta functional is the product of delta functions at each point in spacetime, since both are zero unless the argument is the zero function:

$$\delta[\phi] \sim \prod_x \delta(\phi_x) \ , \tag{6}$$

where $\phi_x = \phi(x)$ is the value of the field at point x (and not a function of x). We also have

$$\delta(\phi_x) = \int \frac{d\omega_x}{2\pi}\, e^{i\omega_x \phi_x} \ . \tag{7}$$

It follows that

$$\begin{aligned} \delta[\phi] &\sim \prod_x \left(\int d\omega_x\, e^{i\omega_x\phi_x}\right) && (8)\\ &\sim \left(\prod_x \int d\omega_x\right) \exp(i \sum_x \omega_x \phi_x) && (9)\\ &\sim \int [\mathscr{D}\omega]\, \exp\left[i \int d^4x\, \omega(x)\phi(x)\right] \ . && (10) \end{aligned}$$

The second kind of integral we will encounter is the Gaussian. Recall that

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \ . \tag{11}$$

The path integral Gaussian is just a multi-dimensional generalization of this. To compute it, it is convenient to think of the set of all functions on spacetime as an infinite-dimensional vector space with an inner product given by

$$f \cdot g = \int d^4x\, f(x)g(x) \ . \tag{12}$$

An element of the space is just a function f, and $f(x)$ is the x-th component of the function.[1]

The integrand of the functional Gaussian integral is

$$\exp\left[-\phi \cdot (\Delta\phi)\right] = \exp\left[-\int d^4x\ \phi(x)\Delta\,\phi(x)\right] \ , \tag{13}$$

where Δ is a differential operator with only even powers of ∂^2. Since ϕ vanishes at the boundary, this operator is Hermitian with respect to the inner product. It follows that we can expand any function ϕ in terms of a set of basis functions $\{f_A\}$ such that

$$\Delta\, f_A = \lambda_A f_A \ , \tag{14}$$

and

$$\int d^4x\ f_A f_B = \delta_{AB} \ . \tag{15}$$

Expanding ϕ in terms of these basis functions gives

$$\phi(x) = \sum_A \phi_A\, f_A(x) \ , \tag{16}$$

where ϕ_A is the A-th expansion coefficient. The Gaussian integrand then becomes

$$\exp\left[-\phi \cdot (\Delta\phi)\right] = \exp\left[-\sum_A \lambda_A \phi_A^2\right] = \prod_A e^{-\lambda_A \phi_A^2} \ . \tag{17}$$

Since any field configuration can be specified completely by the expansion coefficients ϕ_A, integrating over them is equivalent to doing the path integral. Thus, we have

$$\int [\mathscr{D}\phi] \sim \prod_A \left[\int_{-\infty}^{\infty} d\phi_A\right] \ . \tag{18}$$

[1] You've seen this in QM: if $|\psi\rangle$ is a state, $\psi(x) = \langle x|\psi\rangle$ is the x-th component in the position basis.

Putting these pieces together, we see that

$$\begin{aligned}\int [\mathscr{D}\phi]\, e^{-\phi\cdot(\Delta\phi)} &\sim \prod_A \left[\int d\phi_A\, e^{-\lambda_A \phi_A^2}\right] &(19)\\ &\sim \prod_A \left(\frac{\pi}{\lambda_A}\right)^{1/2} &(20)\\ &\sim (detA)^{-1/2}\ . &(21)\end{aligned}$$

Note that defining the determinant of an operator to be the product of its eigenvalues coincides with what we would find for a finite-dimensional diagonalizable matrix.

1.2 Path Integral Quantization of the Free Scalar

Recall that we discussed eigenstates of the Schrödinger-picture field operator $\phi(\vec{x}) = \phi(0,\vec{x})$:

$$\hat{\phi}(\vec{x})|\phi'\rangle = \phi'(\vec{x})|\phi'\rangle\ . \tag{22}$$

On the left-hand side we have the field operator (denoted by the hat), while on the right-hand side we have the specific classical field function that is its eigenvalue. In the Heisenberg picture, we have

$$\hat{\phi}(x) = \hat{\phi}(t,\vec{x}) = e^{iHt}\hat{\phi}(0,\vec{x})e^{-iHt}\ . \tag{23}$$

In this picture, we can also form comoving eigenstates defined by

$$|\phi'(t)\rangle = e^{iHt}|\phi'\rangle\ . \tag{24}$$

These obviously satisfy

$$\hat{\phi}(t,\vec{x})|\phi'(t)\rangle = \phi'(\vec{x})|\phi'(t)\rangle\ . \tag{25}$$

Note that these comoving eigenstates evolve oppositely to physical states in the Schrödinger picture.

The key result that I will state without proof is [1]

$$\langle\phi''(t'')|T\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\}|\phi'(t')\rangle = \int [\mathscr{D}\phi]_{\phi'}^{\phi''} \exp\left[i\int_{t'}^{t''} dt\, L(\phi,\dot{\phi})\right] \phi(x_1)\ldots\phi(x_n)\ , \tag{26}$$

where $t' < t_1, \ldots, t_n < t''$. This is a very deep relation. On the left side is a quantum mechanical matrix element of an operator. On the right, we just have a complicated sum over *classical* field configurations weighted by the action!

For scattering calculations, we only need to find the expectation values of operators in the vacuum state $|\Omega\rangle$ of the theory. As before, we can project it out by taking the limit of

$t \to \pm\infty$ in a slightly imaginary direction. For example,

$$\lim_{t\to-\infty(1-i\epsilon)} |\phi'(t)\rangle = \lim_{t\to-\infty(1-i\epsilon)} e^{iHt'} \left(\sum_\lambda |\lambda\rangle\langle\lambda|\right) |\phi'\rangle \tag{27}$$

$$= \lim_{t\to-\infty(1-i\epsilon)} \left[|\Omega\rangle\langle\Omega|\phi'\rangle + \sum_{\lambda\neq\Omega} e^{iE_\lambda t}|\lambda\rangle\langle\lambda|\phi'\rangle\right] \tag{28}$$

$$= (\langle\Omega|\phi'\rangle)\, |\Omega\rangle \tag{29}$$

Instead of tilting t, we can instead add a small perturbation to the energy to make it slightly imaginary. A convenient way to do this in a field theory is to make the mass slightly imaginary, since the corresponding ϕ^2 operator is positive definite,

$$-\mathscr{L} \supset m^2\phi^2 \to (m^2 - i\epsilon)\phi^2 \ , \tag{30}$$

with $\epsilon > 0$. This induces

$$\mathscr{H}(m^2 - i\epsilon) \to \mathscr{H}(m^2) + \frac{\partial\mathscr{H}}{\partial m^2}(-i\epsilon) = \mathscr{H}(m^2) - i\epsilon' \ , \tag{31}$$

with $\epsilon' > 0$ since $\partial\mathscr{H}/\partial m^2 > 0$ in any sensible theory. This gives

$$E_\lambda \to E_\lambda - i\epsilon' \ , \tag{32}$$

which projects out the vacuum when inserted in Eq. (28).

After applying this projection, we obtain our master formula

$$\langle\Omega|T\{\hat\phi(x_1)\ldots\hat\phi(x_n)\}|\Omega\rangle = \int[\mathscr{D}\phi]\, e^{iS'[\phi]}\, \phi(x_1)\ldots\phi(x_n) \Big/ \int[\mathscr{D}\phi]\, e^{iS'[\phi]} \ , \tag{33}$$

where the S' is the action with a slightly imaginary mass, and we have normalized the right-hand side so that $\langle 1\rangle = 1$. This formula should remind you of a statistical mechanical partition function, with $e^{iS'}$ instead of $e^{-H/T}$ as the exponential weight.

To study operator expectation values, let us define the *generating functional* $Z[J]$ by

$$Z[J] = \int[\mathscr{D}\phi]\, \exp(iS[\phi] + iJ\cdot\phi) = \int[\mathscr{D}\phi]\, \exp\left[i\int d^4x\,(\mathscr{L} + J\phi)\right] \ , \tag{34}$$

where $J(x)$ is an unspecified function and $J\cdot\phi = \int d^4x\, J(x)\phi(x)$ as before. The function $J(x)$ is often called the source. Suppose we take the functional integral of $Z[J]$ with respect to $J(x)$. This has the effect of adding a power of $i\phi(x)$ to the integrand:

$$\frac{\delta Z[J]}{\delta J(x)} = \int[\mathscr{D}\phi]\, \frac{\delta}{\delta J(x)} e^{iS[\phi]+J\cdot\phi} \tag{35}$$

$$= \int[\mathscr{D}\phi]\, \left[\int d^4y\, i\phi(y)\, \delta^{(4)}(y-x)\right]\, e^{iS[\phi]+J\cdot\phi} \tag{36}$$

$$= \int[\mathscr{D}\phi]\, i\phi(x)\, e^{iS[\phi]+J\cdot\phi} \tag{37}$$

Taking more derivatives would add more powers of ϕ. We can use this to rewrite our master formula, Eq. (33), as

$$\langle\Omega|T\{\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\}|\Omega\rangle = (-i)^n\,\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\,Z[J]\bigg|_{J=0}\bigg/\;Z[0]\;. \tag{38}$$

Setting $J\to 0$ after taking the derivatives gets rid of the $J\cdot\phi$ in the exponential, and normalizing by $1/Z[0]$ ensures that $\langle 1\rangle = 1$.

So far, all our results are completely general and apply to both free and interacting scalar theories. Let us now apply them specifically to the free scalar theory,

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(m^2 - i\epsilon)\phi^2\;. \tag{39}$$

after integrating by parts, this gives

$$\begin{aligned} S[\phi] &= \frac{1}{2}\int d^4x\;\phi(-\partial^2 - m^2 + i\epsilon)\phi && (40)\\ &:= \frac{1}{2}\,\phi\cdot(\Delta\phi)\;, && (41)\end{aligned}$$

with $\Delta = (-\partial^2 - m^2 + i\epsilon)$. When inserted into the path integral, this will just be a Gaussian which we know how to evaluate.

We can evaluate the generating functional for the free theory $Z_0[J]$ explicitly by completing the square and computing the Gaussian path integral. For this, note that

$$\begin{aligned} \Delta\;D_F(x) &= (-\partial^2 - m^2 + i\epsilon)\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 + i\epsilon}\,e^{-ip\cdot x} && (42)\\ &= i\delta^{(4)}(x)\;. && (43)\end{aligned}$$

We can therefore think of $-iD_F = \Delta^{-1}$. The trick for evaluating is to make the change of variable

$$\phi(x) = \phi'(x) - \int d^4y\,(-i)D_F(x-y)J(y) := \phi'(x) - (\Delta^{-1}J)(x)\;. \tag{44}$$

In the exponential of $Z[J]$, this gives

$$\begin{aligned} S[\phi] + J\cdot\phi &= \frac{1}{2}(\phi' - \Delta^{-1}J)\cdot[\Delta(\phi' - \Delta^{-1}J)]\;+\;J\cdot(\phi' - \Delta^{-1}J) && (45)\\ &= \frac{1}{2}\,\phi'\cdot(\Delta\phi') - \frac{1}{2}\,J\cdot(\Delta^{-1}J) && (46)\\ &= \frac{1}{2}\int d^4x\,\phi'\Delta\phi' + \frac{i}{2}\int d^4x\int d^4y\;J(x)D_F(x-y)J(y)\;. && (47)\end{aligned}$$

In going from the first to the second line we have implicitly integrated by parts to move the Δ operator around. The integrand in terms of ϕ' is now a Gaussian. Since the change of variables was just a shift by a constant, the path integral measure is also unchanged,

$$[\mathscr{D}\phi] = [\mathscr{D}\phi']\;. \tag{48}$$

It follows that the free-theory generating functional is equal to

$$Z_0[J] \sim (\det \Delta)^{-1/2} \exp\left[-\frac{1}{2}\int d^4x \int d^4y \; J(x) D_F(x-y) J(y)\right] . \tag{49}$$

Note that the $i\epsilon$ factor inserted to project out the ground state leads to the Feynman propagator in this expression. It also ensures the convergence of the Gaussian integral.

We can now use Eq. (38) to compute n-point functions in the free theory. The 2-point functions comes out to be

$$(-i)^2 \frac{\delta^2}{\delta J(x)\, \delta J(y)} \left(e^{-\frac{1}{2}J\cdot D_F \cdot J}\right)\Big|_{J=0} = (-1)^2 D_F(x-y) \; , \tag{50}$$

which is precisely what we found previously. We also see that the propagator really is the inverse of the quadratic operator in the Lagrangian. Repeating with more derivatives, we reproduce just Wick's theorem in a very trivial way. In this case, the contraction of two fields corresponds to the corresponding derivatives hitting the same term in the expansion of the exponential.

1.3 Path Integrals for Fermions

We can also quantize fermion fields using path integrals. For this, we need to define regular derivatives and integrals of Grassmann numbers. Suppose we have a single Grassmann number η. Since $\eta^2 = 0$, the most general real function of η is

$$f(\eta) = a + b\eta \; , \tag{51}$$

for some real coefficients a and b. We define the derivative by

$$\frac{d}{d\eta} 1 = 0, \qquad \frac{d}{d\eta}\eta = 1 \; , \tag{52}$$

so that

$$\frac{d}{d\eta} f(\eta) = b \; . \tag{53}$$

For integrals, we define

$$\int d\eta\, 1 = 0, \qquad \int d\eta\, \eta = 1 \; . \tag{54}$$

It is best to think of $\int d\eta$ as an operator on functions of η rather than an integral in the usual sense. This definition implies that

$$\int d\eta \, \frac{df(\eta)}{d\eta} = 0 \; . \tag{55}$$

When there are multiple Grassmann numbers, we treat the integral and the derivative as operators acting from the left, and put everything together by anticommuting. For example, given η and χ we have

$$\frac{d}{d\eta}\chi = 0, \qquad \frac{d}{d\eta}(\chi\eta) = -\frac{d}{d\eta}(\eta\chi) = -\chi \ . \tag{56}$$

Integrals work in the same way.

Having warmed up with regular derivatives and integrals, let's jump right into functional derivatives and integrals. We have

$$\frac{\delta\Psi_a(x)}{\delta\Psi_b(y)} = \delta_a{}^b\,\delta^{(4)}(x-y) \ . \tag{57}$$

For the functional integral, we take

$$\int[\mathscr{D}\Psi] \sim \prod_x \left[\int d\Psi_1(x)\int d\Psi_2(x)\int d\Psi_3(x)\int d\Psi_4(x)\right] \ . \tag{58}$$

In other words, do the Grassmann integral over each component of Ψ at every point in spacetime and multiply together all the results. Note that $\Psi_a(x)$ should be treated as independent Grassmann variables in that $\Psi_a(x)\Psi_b(x')$ is non-zero unless $a = b$ and $x = x'$.

Evaluating fermionic functional integrals is even easier than for bosons. The case of most interest to us will the be the Gaussian,

$$\int[\mathscr{D}\Psi\,\mathscr{D}\bar{\Psi}]\ \exp\left[-\int d^4x\,\bar{\Psi}\Delta\Psi\right] \ , \tag{59}$$

where Δ is a Hermitian differential operator. As before, let us expand Ψ and $\bar{\Psi}$ in orthonormal eigenfunctions of the operator Δ:

$$\Psi(x) = \sum_A \psi_A g_A(x), \qquad \bar{\Psi}(x) = \sum_B \bar{\psi}_B h_B(x) \ , \tag{60}$$

where g_A and h_A are bosonic functions and ψ_A and $\bar{\psi}_B$ are Grassmannian expansion coefficients. For the eigenfunctions, we have

$$\Delta\, g_A = \lambda_A g_A \qquad \Delta\, h_B = \lambda_B h_B \ , \tag{61}$$

as well as

$$\int d^4x\, h_B(x) g_A(x) = \delta_{AB} \ . \tag{62}$$

Putting this into the integrand, we find

$$\exp\left[\bar{\Psi}\cdot(\Delta\Psi)\right] = \exp\left[-\sum_A \bar{\psi}_A\lambda_A\psi_A\right] = \prod_A e^{-\bar{\psi}_A\lambda_A\psi_A} \ . \tag{63}$$

The path integral measure becomes

$$\int [\mathscr{D}\Psi\, \mathscr{D}\bar{\Psi}] \sim \prod_A \left[\int d\psi_A \int d\bar{\psi}_A \right] . \tag{64}$$

The Gaussian integral is therefore equal to

$$\begin{aligned} \int [\mathscr{D}\Psi\, \mathscr{D}\bar{\Psi}]\, e^{-\bar{\Psi}\cdot(\Delta\Psi)} &\sim \prod_A \left[\int d\psi_A \int d\bar{\psi}_A\, e^{-\bar{\psi}_A \lambda_A \psi_A} \right] && (65)\\ &\sim \prod_A \left[\int d\psi_A \int d\bar{\psi}_A\, (1 - \lambda_A \bar{\psi}_A \psi_A) \right] && (66)\\ &\sim \prod_A (-\lambda_A) && (67)\\ &\sim \det(\Delta) . && (68) \end{aligned}$$

Note that the determinant has a positive power rather than a negative one. This is characteristic of fermions. The factor of +1 relative to 1/2 comes from the fact that two different fields are now involved.

With these tools in hand, we can now turn to physics. The master formula for a Dirac fermion is

$$\langle \Omega | T\{\hat{\Psi}_{a_1}(x_1) \ldots \hat{\bar{\Psi}}^{b_n}(x_n)\} | \Omega \rangle = \int [\mathscr{D}\Psi\, \mathscr{D}\bar{\Psi}]\, e^{iS'}\, \Psi_{a_1}(x_1) \ldots \bar{\Psi}^{b_n}(x_n) \Big/ \int [\mathscr{D}\Psi\, \mathscr{D}\bar{\Psi}]\, e^{iS'}, \tag{69}$$

where $S' = S[\Psi, \bar{\Psi}] + i\epsilon \bar{\Psi}\Psi$. When there are multiple types of fields, we should integrate over each one of them in the path integral.

To compute n-point functions, it is again convenient to define a generating functional, this time with a source for each independent field. For a theory of Dirac fermions,

$$Z[\eta, \bar{\eta}] = \int [\mathscr{D}\Psi \mathscr{D}\bar{\Psi}]\, \exp\left[iS[\Psi, \bar{\Psi}] + i\bar{\eta} \cdot \Psi + i\Psi \cdot \eta \right] . \tag{70}$$

We now have

$$\begin{aligned} \frac{\delta Z}{\delta \bar{\eta}^a(x)} &= \int [\mathscr{D}\Psi \mathscr{D}\bar{\Psi}]\, (-i\Psi_a(x))\, e^{iS[\Psi,\bar{\Psi}] + i\bar{\eta}\cdot\Psi + i\Psi\cdot\eta} , && (71)\\ \frac{\delta Z}{\delta \eta_b(x)} &= \int [\mathscr{D}\Psi \mathscr{D}\bar{\Psi}]\, (i\bar{\Psi}^b(x))\, e^{iS[\Psi,\bar{\Psi}] + i\bar{\eta}\cdot\Psi + i\Psi\cdot\eta} . && (72) \end{aligned}$$

Note the extra minus sign from anticommuting the fermionic functional derivative. Taking multiple derivatives and setting $\eta = \bar{\eta} = 0$, we get

$$\langle \Omega | T\{\Psi_{a_1}(x_1) \ldots \bar{\Psi}^{b_n}(x_n)\} | \Omega \rangle = \frac{1}{Z[0,0]} \left(-i \frac{\delta}{\delta \bar{\eta}^{a_1}(x_1)} \right) \ldots \left(+i \frac{\delta}{\delta \eta_{b_n}(x_n)} \right) Z[\eta, \bar{\eta}] \Big|_{\eta = \bar{\eta} = 0} \tag{73}$$

Let's now specialize to the free theory of a Dirac fermion, with action

$$S[\Psi, \bar{\Psi}] = \int d^4x\, \bar{\Psi}(i\gamma \cdot \partial - m + i\epsilon)\Psi := \bar{\Psi} \cdot (\Delta\Psi) . \tag{74}$$

Just like before, we can complete the square in the generating functional $Z_0[\eta, \bar{\eta}]$ and do the resulting Gaussian integral. The appropriate changes of variables in this case are

$$\Psi(x) = \Psi'(x) - \int d^4y \, (-i) S_F(x-y) \eta(y) \tag{75}$$

$$\bar{\Psi}(x) = \bar{\Psi}'(x) - \int d^4y \, (-i) \bar{\eta}(y) S_F(y-x) \, . \tag{76}$$

Using the fact that

$$\Delta S_F(x) = i\delta^{(4)}(x) \, , \tag{77}$$

we obtain

$$Z_0[\eta, \bar{\eta}] \sim \det(\Delta) \, \exp\left[- \int d^4x \int d^4y \, \bar{\eta}(x) S_F(x-y) \eta(y) \right] \, . \tag{78}$$

This reproduces Wick's theorem for Dirac fermions.

1.4 Interacting Theories

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2 Quantizing the Photon

We turn next to quantizing the free photon,

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \, . \tag{79}$$

Path integrals will give us a nice way to remove the gauge redundancy.

2.1 Degrees of Freedom: Massive Vector

Let's start by counting degrees of freedom in vector fields. Before tackling the photon, let's study general, massive vectors. A generic vector field A^μ has four real degrees of freedom. However, a spin $s = 1$ state only has three degrees of freedom. What is happening is that the generic vector decomposes into and $s = 0$ and $s = 1$ components under the rotation subgroup of the Lorentz group. The $s = 0$ piece can be thought of as the subset of A^μ fields that can be written as $A_\mu = \partial_\mu \phi$ for some scalar ϕ.

We would like to formulate a Lagrangian for a massive vector that eliminates the scalar component of A^μ from the dynamics in a Lorentz-invariant way. The trick to this is to impose the constraint $\partial_\mu A^\mu = 0$ and to take the Lagrangian to be

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \tag{80}$$

$$= \frac{1}{2} A^\mu (\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^\nu + \frac{1}{2} m^2 A_\mu A^\mu \, , \tag{81}$$

where we have dropped the total derivative in the second line that will vanish when inserted into the action. This kinetic term matches what we had for the photon. It looks funny, but the essential feature is that it annihilates the scalar component of A^μ,

$$(\eta^{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\partial^\mu\phi = 0 \ . \tag{82}$$

Therefore no kinetic term is induced for ϕ.

It is also possible to couple the massive vector to other stuff. If we were to write an interaction for A^μ that is linear in the field, it would have to take the form

$$\mathscr{L} \supset A_\mu j^\mu \ , \tag{83}$$

for some four-vector operator j^μ. We would like to avoid having the coupling source $A_\mu = \partial_\mu\phi$. Plugging this form into Eq. (83), we find

$$\mathscr{L} \supset \partial_\mu\phi\, j^\mu = -\phi\,(\partial_\mu j^\mu) \ , \tag{84}$$

up to a total derivative. This implies that the scalar component will not be sourced provided the operator j^μ is a conserved current.

We now see how to consistently write a Lorentz-invariant theory that only keeps the $s = 1$ part of A^μ. Using the free Lagrangian of Eq. (81) and possibly with an interaction of the form of Eq. (83), we find that the scalar part of A^μ has no kinetic term and does not couple to anything through the interaction. The only equation we have for it comes from the constraint, which implies

$$\partial^2\phi = 0 \ . \tag{85}$$

The only solution to this that vanishes at infinity is $\phi = 0$. Thus, we have eliminated the scalar part. For the non-scalar part, the equation of motion is

$$(\partial^2 + m^2)A^\mu = 0 \ . \tag{86}$$

This is just the thing we want.

2.2 Degrees of Freedom: Massless Vector

We turn next to the free photon. The Lagrangian can be rewritten as

$$\mathscr{L} \ = \ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}A^\mu(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu \ , \tag{87}$$

where we have again dropped a total derivative. Unlike the massive vector, there is additional constraint. However, we do have a new gauge invariance under

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha \ , \tag{88}$$

for any smooth function α. The interpretation is also different: any two field configurations related by a gauge transformation are physically equivalent.

The equations of motion are

$$0 = \partial^2 A^0 - \partial_0(\partial_0 A^0 + \partial_i A^i) \tag{89}$$
$$0 = -\partial^2 A^i - \partial_i(\partial_0 A^0 + \partial_j A^j) \ , \tag{90}$$

where we have written the time and space components separately. Before trying to solve these equations, it is helpful to choose a specific gauge to make it easier to solve them. A popular choice is the *Coulomb gauge*, $\partial_i A^i = \vec{\nabla} \cdot \vec{A} = 0$. Starting in gauge in which this is not satisfied, we can always make a gauge transformation such that it is true for the transformed fields,

$$0 = \partial_i A^{i'} = \partial_i A^i + \frac{1}{e}\partial_i^2 \alpha \ . \tag{91}$$

This almost completely fixes α, but it does leave a residual gauge freedom to transform by any α such that $\partial_i^2 \alpha = 0$. With the Coulomb gauge choice, A^0 satisfies

$$0 = \partial_i^2 A^0 \ . \tag{92}$$

The important part about his relation is that there are no time derivatives. The only solution that vanishes at spatial infinity is $A^0 = 0$. Plugging back into the equation for A^i, we see that

$$\partial^2 A^i = 0 \tag{93}$$

which is the usual wave equation.

Suppose we look for wave solutions, of the form

$$A^\mu = \epsilon^\mu(k, \lambda) e^{ik\cdot x} \ . \tag{94}$$

The equation of motion forces $k^2 = 0$, while the gauge condition implies

$$k_i \epsilon^i = 0, \qquad \epsilon^0 = 0 \ . \tag{95}$$

These two conditions allow two independent polarization vectors.[2] This is just the right number to describe a massless particle; gauge invariance has removed the unwanted degrees of freedom! The most general solution is

$$A^\mu = \sum_{\lambda=1}^{2} \int \widetilde{dk} \ \left[a(\vec{k}) \epsilon^\mu(k, \lambda) e^{-ik\cdot x} + a^*(\vec{k}) \epsilon^{\mu *}(k, \lambda) e^{ik\cdot x} \right] \ , \tag{96}$$

where the polarizations satisify Eq. (95).

[2] For $k = (k, 0, 0, k)$, they can be chosen to be $\epsilon^\mu(k, 1) = (0, 1, 0, 0)$ and $\epsilon^\mu(k, 2) = (0, 0, 1, 0)$.

2.3 Quantizing the Free Photon

Applying the usual techniques to quantizing the free photon, we run into some challenges related to gauge invariance. In the canonical picture, the first problem is that the momentum conjugate to A^0 vanishes,

$$\Pi^\nu = \frac{\partial \mathscr{L}}{\partial(\partial_0 A_\nu)} = -(\partial^0 A^\nu - \partial^\nu A^0) \ , \qquad (97)$$

which is clearly zero for $\nu = 0$. This challenge can be overcome by working in the Coulomb gauge, which removes A^0 as a dynamical degree of freedom. However, maintaining the Coulomb gauge constraint can be cumbersome.

Instead of going the canonical route, let us try working with path integrals. At first glance, things look even worse. In our previous examples, we found that the propagator was given by the inverse of the quadratic operator in the Lagrangian. Here the quadratic term is given by Eq. (87). In momentum space, this operator becomes

$$P_{\mu\nu} = (-\eta_{\mu\nu} k^2 + k_\mu k_\nu) \ . \qquad (98)$$

The problem is that this operator has zero eigenvalues, $P_{\mu\nu} k^\nu = 0$ (equivalent to the annihilation of $\partial_\mu \phi$ in position space). Therefore this operator is not invertible, and our propagator would seem to be ill-defined.

The second challenge to path-integral quantization is that the sum over field configurations will be highly redundant. We expect to have path integrals like

$$\int [\mathscr{D} A_\mu] \, \mathcal{O}(A) \, e^{iS[A]} \ , \qquad (99)$$

where $\mathcal{O}(A)$ is some operator built out of A_μ fields. In integrating over configurations of A^μ, we will end up summing over many configurations related by gauge transformations that are physically equivalent. This will not be so bad if the operator is also gauge invariant, since the redundant integrations will just produce an overall pre-factor related to the formally infinite volume of the gauge invariance. Our strategy will be to find a way to factor out the volume. It turns out that this will also fix the propagator inversion problem.

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