PHYS 526 Notes #0: Notation and Background

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1 Notational Conventions

A large fraction of this course will deal with highly relativistic systems. For this reason, we will use a notation and a set of units that is geared to this situation [**?**, **?**, **?**].

1.1 Natural Units

We will express quantities in so-called *natural units*, defined by

$$\hbar = c = 1 \ , \tag{1}$$

where $\hbar$ is the usual quantum mechanics thing and c is the speed of light. Since $\hbar$ has units of energy times time, $\hbar = 1$ implies that we are measuring time in units of inverse energy. Similarly, $c = 1$ means we are measuring distance in units of time and therefore units of inverse energy as well. This simplifies dimensional analysis since now all dimensionful quantities can be expressed in units of energy. For example,

$$[E] = [P] = [M] = +1, \quad [L] = [T] = -1, \quad \left[\frac{d}{dx}\right] = +1, \tag{2}$$

where the square brackets denotes the energy dimension of the quantity (in natural units, of course). The specific unit we will use for energy is the electron Volt (eV), corresponding to the energy acquired by an electron passing through a potential difference of one Volt. We will also use $\text{keV} = 10^3\,\text{eV}$, $\text{MeV} = 10^6\,\text{eV}$, $\text{GeV} = 10^9\,\text{eV}$, and $\text{TeV} = 10^{12}\,\text{eV}$.

To put a result back into regular units, just add powers of of $\hbar$ and c ($\sim L/T$) until you get what you want. In doing so, it's handy to remember two things:

$$\hbar\, c \ = \ 1 \ \simeq \ 0.197\,\text{GeV}\cdot\text{fm} \ , \tag{3}$$

$$c \ = \ 1 \ \simeq \ 3.0 \times 10^{10}\,\text{cm} \ , \tag{4}$$

where $1\,\text{fm} = 10^{-13}\text{cm}$. Despite our usage of natural units, scattering cross sections (which have units of area) will often be expressed in barns (b), with $1\,\text{b} = 10^{-24}\text{cm}^2$.

1.2 Index Notation

We will mostly study systems that are invariant under special relativity, meaning that the underlying equations of the system take the same form after applying Lorentz transformations (boosts and rotations) to the system (which we will get to in more detail later on).

Discussing such systems is much easier if we use an index notation. Instead of writing (t, x, y, z) for a specific point in space and time, we will use

$$x^\mu \;=\; (t, x, y, z), \quad \mu = 0, 1, 2, 3 \; . \tag{5}$$

We will call this a (position) four-vector. It is useful because the components of x^μ transform into each other under Lorentz transformations. Similarly, we will combine energy and momentum into a (momentum) four-vector

$$p^\mu \;=\; (E, p^x, p^y, p^z) \; . \tag{6}$$

For any four-vector $a^\mu = (a^0, a^1, a^2, a^3)$, we define a vector with a lowered index by

$$a_\mu \;=\; \eta_{\mu\nu} a^\nu \; , \tag{7}$$

where the two-index object $n_{\mu\nu}$ has components

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \; . \tag{8}$$

In writing Eq. (**??**), we have also used Einstein's summation convention, where we implicitly sum over repeated indices[1]. In full gory detail, Eq.(**??**) is equal to

$$\begin{aligned} a_\mu &= \eta_{\mu\nu} a^\nu && (9) \\ &= \sum_{\nu=0}^{4} \eta_{\mu\nu} a^\nu && (10) \\ &= \begin{cases} a^0 & ; \quad \mu = 0 \\ -a^i & ; \quad \mu = i = 1, 2, 3 \end{cases} && (11) \end{aligned}$$

It is really important to keep upper and lower indices distinct because it will turn out that they refer to different transformation properties under Lorentz. It is also convential to use Greek letters for the indices of four-vectors and Roman letters for the components of three-vectors.

Since $\eta_{\mu\nu}$ is invertible, we can also raise indices with the inverse matrix $\eta^{\mu\nu}$:

$$a^\mu \;=\; \eta^{\mu\nu} a_\nu = \eta^{\mu\nu} (\eta_{\nu\kappa} a^\kappa) = \delta^\mu{}_\kappa a^\kappa = a^\mu \; , \tag{12}$$

where we have used the definition of the inverse matrix, and $\delta^\mu{}_\nu$ is the unit matrix, equal to one for $\mu = \nu$ and zero otherwise. Note, however, that in terms of components $\eta_{\mu\nu} = \eta^{\mu\nu}$.

Combining the summation convention with raising and lowering, we can define a four-vector dot-product by

$$\begin{aligned} a \cdot b &= \eta_{\mu\nu} a^\mu b^\nu && (13) \\ &= a_\nu b^\nu && (14) \\ &= a^0 b^0 - \vec{a} \cdot \vec{b} \; . && (15) \end{aligned}$$

[1] Apparently Einstein considered this his greatest contribution to physics

It will turn out that a quantity is invariant under Lorentz transformations if all the indices in it are contracted. A familiar example is the square of the momentum four-vector of a particle:

$$p^2 = E^2 - \vec{p}^2 = m^2 \ , \tag{16}$$

where m is the (rest) mass of that particle.

We will also use a simplified index notation for derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{17}$$

$$\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2 - \vec{\nabla}^2 \tag{18}$$

This implies, for example, that

$$\partial_\mu x^\nu = \delta_\mu^\nu \ . \tag{19}$$

Note that we treat t and $\vec{x}$ as independent variables.

2 Background

Having established notation, we turn next to a quick review of the essential background for this course. If you are not familiar with this material, you should come speak to me as soon as possible.

2.1 Electromagnetism

Maxwell's equations [**?**]:

$$\vec{\nabla}\cdot\vec{E} = \rho \tag{20}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{21}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{22}$$

$$\vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{j} \tag{23}$$

where ρ is the charge density and $\vec{j}$ is the charge current. In writing the equations in this way, we have implicitly also set $\epsilon_0 = \mu_0 = 1$ (consistent with $c = 1/\sqrt{\epsilon_0\mu_0}$).

The electric and magnetic fields can be written in terms of electric and magnetic potentials:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla}\times\vec{A} \ . \tag{24}$$

Down the road, we will combine the electric and magnetic potentials into a single four-vector

$$A^\mu = (\phi,\, \vec{A}) \ . \tag{25}$$

Note that with these conventions, the electric potential due to a stationary point charge is

$$\phi = \frac{Q}{4\pi}\frac{1}{r} \ . \tag{26}$$

2.2 Classical Mechanics

The equations of motion for a system described by the generalized coordinates $q_i(t)$ can be derived from an action formed from a Lagrangian [**?**]:

$$S = \int dt\ L(q_i, \dot{q}_i) \ . \tag{27}$$

In many cases, the Lagragian L is equal to $L = T - V$. The equations of motion can be derived from the principle of stationary action, namely that the physical configuration is the one that satisfies $\delta S = 0$ for any infinitessimal variation $q_i(t) \to q_i(t) + \delta q_i(t)$. Applying this condition gives the equations of motion

$$\partial_t \left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \ . \tag{28}$$

for each value of i.

A second way to formulate classical mechanics is in terms of a Hamiltonian H that depends on the coordinates $q_i(t)$ and $p_i(t)$ [**?**]. Given a Lagrangian $L(q_i, \dot{q}_i)$, the generalized momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \ . \tag{29}$$

Using these equations, it should be possible to rewrite all of the $\dot{q}_i$ in terms of the p_i (and possibly q_i). The Hamiltonian $H(q_i, p_i)$ is given by

$$H = \sum_i \dot{q}_i p_i - L \ , \tag{30}$$

where all the $\dot{q}_i$s on the right-hand side are to be seen as functions of the q_i and p_i. The equations of motion in this formulation can be written in terms of Poisson brackets, defined according to

$$\{f,\, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial f}{\partial q_i}\right) \ . \tag{31}$$

With this definition, the equations of motion are

$$\dot{q}_i = \{q_i,\, H\}, \qquad \dot{p}_i = \{p_i,\, H\} \ . \tag{32}$$

Two useful facts to keep in mind are

$$\{q_i,\, p_j\} = \delta_{ij} \ , \tag{33}$$

and for any function $f(q, p, t)$

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \ . \tag{34}$$

2.3 Quantum Mechanics

The rules of quantum mechanics (QM) for relativistic systems are exactly the same as what you have learned in previous classes. However, what we will do with these rules may seem slightly different from what you have done in the past. In the QM courses you have taken, you were most likely given a Hamiltonian for a system and asked to solve for the corresponding eigenvalues and eigenstates. Instead, the major challenge to be addressed in this course is how to construct a sensible Hamiltonian to describe electromagnetism (or relativistic particles). So, even though the way we approach things might seem new and unusual, keep in mind that the basic rules are the same as ever.

Recall that a quantum mechanical system consists of a set of states (comprising a Hilbert space) together with the set of operators acting on them [**?**]. Starting from a classical system defined in terms of q_i, p_j, and H, we can construct the corresponding quantum system by elevating q_i and p_j to operators on the Hilbert space satisfying

$$[q_i,\, p_j] = i\,\delta_{ij} \ . \tag{35}$$

Notice the similarity to the Poisson brackets. The Hamiltonian H is just a function of the q_i and p_j, and is now also a well-defined operator on the Hilbert space of states (up to possible ambiguities in the order of the q_i and p_j).

The Hamiltonian determines the time evolution of the system. There are two popular ways to describe this. The first is the Schrödinger picture in which the operators are time-independent while a given state evolves according to

$$|\alpha(t)\rangle = e^{-iH(t-t_0)}|\alpha(t_0)\rangle \ , \tag{36}$$

where t_0 is some reference time. Equivalently, states satisfy the Schrödinger equation,

$$\frac{d}{dt}|\alpha(t)\rangle = -iH|\alpha(t)\rangle \ . \tag{37}$$

The second standard way to describe time evolution in QM is the Heisenberg picture, in which states are time independent while the operators depend on time. Specifically, we have $|\alpha(t)\rangle = |\alpha(t_0)\rangle$ together with

$$\mathcal{O}(t) = e^{iH(t-t_0)}\mathcal{O}(t_0)e^{-iH(t-t_0)} \ . \tag{38}$$

This relation is equivalent to the commutation relation

$$\frac{d}{dt}\mathcal{O} = i[\mathcal{O}, H] \ , \tag{39}$$

which should remind you of the classical Poisson bracket result.

This discussion should all be very familiar from your previous courses on QM, so let's try something a bit different. Specifically, let's build a Hilbert space for a system and figure out what kinds of operators act on it.

e.g. 1. Particles in Zero Dimensions

Consider a system consisting of a box that can contain any number of identical (bosonic), non-interacting particles at rest. Let us assume further that each particle contributes an energy M to the total. A complete set of states is therefore $\{|n\rangle \ : \ n = 0, 1, 2, \ldots\}$, where n labels the number of particles in the box. The total energy of state $|n\rangle$ is $n\,M$, so we have

$$H|n\rangle = nM|n\rangle \ .$$

At this point, we are basically done. We have a complete Hilbert space for the system and we can figure out the time evolution of any observable by applying the Schrödinger equation.

Even so, let us fiddle a bit more and look into what sorts of operators can act on the space. Just about the only thing we can do to the system is add or remove particles, so it's worth thinking about operators that would accomplish this. Let us *define* an operator that removes particles one at a time by

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \ .$$

There is nothing stopping us from making such a definition; this is a perfectly well-defined operator on the Hilbert space. Taking the Hermitian conjugate of this and contracting the result with all possible kets, we also find

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ .$$

From this, we can also derive

$$[a, a^\dagger] = 1 \ ,$$

as well as

$$H = M\,a^\dagger a \ .$$

In fact, it can be shown that any operator whatsoever on the Hilbert space can be built up from sums and products of a and $a^\dagger$

All this should be familiar from the case of the harmonic oscillater. Now, however, the system does not correspond to a single point particle in a quadratic potential. Even so, we

can rewrite the Hamiltonian in terms of X and P operators. For this, let us take the two linear combinations of a and $a^\dagger$ that are Hermitian:

$$X = (a + a^\dagger), \qquad P = -i(a - a^\dagger) \ .$$

These relations are invertible, so any operator on the Hilbert space can be constructed from X and P as well. In particular, we have

$$H = M(X^2 + P^2)/2 - M/2 \ . \tag{40}$$

Thus, our system of identical stationary bosons can be described by continuous generalized coordinate operators X and P that don't have anything to do with space or velocity.

3 Useful Math

The mathematics to be used in this course will consist mostly of vector calculus, together with a bit of group theory. We will get to the group theory later on, so let us just mention a few useful results in vector calculus [**?**].

We will write n-dimensional integrals over spacetime (one time and $(n-1)$ spatial dimensions) as

$$\int d^n x = \int dt \int dx^1 \int dx^2 \ldots \int dx^{n-1} \tag{41}$$

The divergence theorem generalizes to this case: for any n-dimensional volume V with boundary ∂V, we have

$$\int_V d^n x \, \partial_\mu f^\mu = \int_{\partial V} dA \ n_\mu f^\mu \ , \tag{42}$$

where n_μ is the outwardly pointing normal unit vector to the bounding surface ∂V. In most cases of interest to us, the surface integral over ∂V will be zero for one reason or another.

Multi-dimensional delta functions will appear frequently:

$$\int d^n x \ \delta^{(n)}(x - x') f(x) = f(x') \ . \tag{43}$$

Equivalently $\delta^{(n)}(x - x') = \prod_{\mu=0}^{n-1} \delta(x^\mu - x^{\mu'})$. We also have the Fourier transform relation

$$\int d^n x \, e^{ik\cdot x} = (2\pi)^n \delta^{(n)}(k), \qquad \int d^n k \, e^{ik\cdot x} = (2\pi)^n \delta^{(n)}(x) \ . \tag{44}$$

Finally, recall that

$$\int dx \ \delta(f(x)) \, g(x) = \sum_a g(x_a)/|df/dx|_{x_a} \ , \tag{45}$$

where x_a refers to any value of x such that $f(x_a) = 0$.

Sometimes it will be useful to use contour integration in the complex plane. The key result is Cauchy's theorem, which is that for any integral around a closed counterclockwise contour C in the complex plane,

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)} \ , \tag{46}$$

provided $f(z)$ has no poles in the region bounded by the contour. If the contour runs clockwise, the formula picks up an overall minus sign. Differentiating both sides with respect to z_0 gives

$$\frac{d^n}{dz^n} f(z_0) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)^{n+1}} \ . \tag{47}$$

This is really useful because it also applies to any deformation of the contour C that does not intersect the pole. Sometimes the stuff in the integrand (with the pole factored off) is called the residue of the pole. Cauchy's theorem is therefore often stated as

$$\oint_C dz \, f(z) = 2\pi i \times (\text{sum of residues}) \ . \tag{48}$$

e.g. 2. $I = \int_0^\infty 1/(1 + x^2)$

We can do this integral easily using Cauchy's theorem. See Fig. **??** for an illustration of the contours C_1 and C_2.

$$\begin{aligned}
I &= \frac{1}{2} \int_{-\infty}^{\infty} dx \ \frac{1}{1 + x^2} \\
&= \frac{1}{2} \lim_{R \to \infty} \int_{-R}^{+R} dx \ \frac{1}{1 + x^2} = \lim_{R \to \infty} \int_{C_1} dz \ \frac{1}{1 + z^2} \\
&= \frac{1}{2} \lim_{R \to \infty} \left[\oint_{C = C_1 + C_2} dz \ \frac{1}{(z + i)(z - i)} - \int_{C_2} dz \ \frac{1}{1 + z^2} \right] \\
&= \frac{1}{2} \times 2\pi i \times \frac{1}{2i} \ - \ 0 \\
&= = \frac{\pi}{2} \ .
\end{aligned}$$

In the last line, we have used the fact that the contour integration along C_2 can be parametrized by $z = Re^{i\theta}$ and goes like

$$\lim_{R \to \infty} \int_{C_2} dz \ \frac{1}{1 + z^2} = \lim_{R \to \infty} \int d\theta \, iRe^{i\theta} \frac{1}{1 + R^2 e^{2i\theta}} \to 0 \ .$$

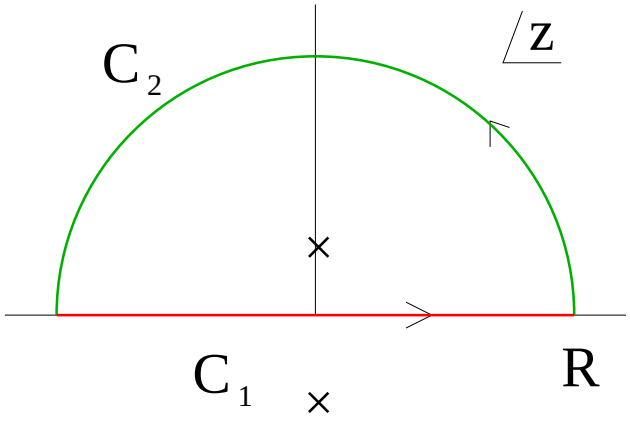

Figure 1: Contours for *e.g. 2*.

One last useful thing to go over is matrix multiplication in index notation. Given an n-dimensional column vector v, we can write its components as v_i:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \ . \tag{49}$$

In this form, we can rewrite the product of a row vector and column vector as

$$u^t v = (u_1, u_2, \ldots, u_n) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \sum_i u_i v_i \ . \tag{50}$$

Similarly, given an $n \times n$ matrix M with elements M_{ij}, we can write the product of the matrix with a column vector (which itself is a column vector) as

$$(M\, v)_i = \sum_j M_{ij} v_j \ . \tag{51}$$

The product of two matrices M and N (which is a matrix MN with elements $(MN)_{ij}$) is simply

$$(MN)_{ij} = \sum_k M_{ik} N_{kj} = \sum_\ell M_{i\ell} N_{\ell j} \ . \tag{52}$$

Note that the i and j indices are fixed, and must match up on both sides of the equation. On the other hand, the k index that is summed over is a dummy index, and can be called anything you like as we have illustrated above.

For three-vectors, we have a cross product operation that takes a pair of vectors and makes another. It can be written in terms of indices using the antisymmetric tensor ϵ_{ijk} with

$$\epsilon_{123} = +1 = \epsilon_{231} = \epsilon_{312} \ , \tag{53}$$

$$\epsilon_{132} = -1 = \epsilon_{213} = \epsilon_{321} \ , \tag{54}$$

and all other entries equal to zero. This gives

$$(a \times b)_i = \sum_{j,k} \epsilon_{ijk} a_j b_k \ . \tag{55}$$

If you are not convinced by this, work out the components explicitly.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[2] M. Srednicki, "Quantum field theory," *Cambridge, UK: Univ. Pr. (2007) 641 p*

[3] L. H. Ryder, "Quantum Field Theory," *Cambridge, Uk: Univ. Pr. (1985) 443p*

[4] D. J. Griffiths, "Introduction to Electrodynamics", *Saddle River, USA: Prentice Hall (1999)*

[5] H. Goldstein, "Classical Mechanics", *Cambridge USA: Addison-Wesley (1950)*

[6] D. J. Griffiths, "Introduction to Quantum Mechanics", *Saddle River, USA: Prentice Hall (2004)*

[7] J. Mathews, R. L. Walker, "Mathematical Methods of Physics", *Redwood City, USA: Addison Wesley (1970)*