A Family Symmetry that Yields the Measured Reactor Angle

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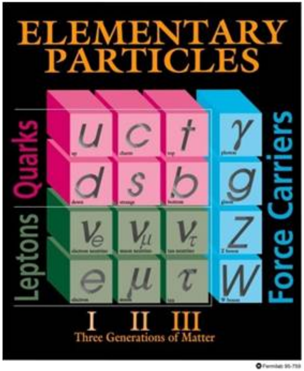
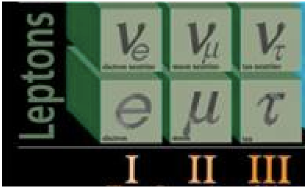
CONTENTS

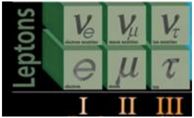
I

Introduction



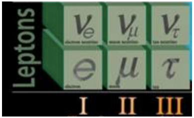


Is there a finite family symmetry consistent with the neutrino mixing data?



electroweak

Particles	$[SU(2), U(1)_Y]$	$\to U(1)_Q$	
$\nu_a := (\nu_e, \nu_\mu, \nu_\tau)$ $N_a := (N_e, N_\mu, N_\tau)$	$[2, -1]$ $[1, \ \ 0]$	Q Q	.
$e_a := (e_L, \mu_L, \tau_L)$ $E_a := (e_R, \mu_R, \tau_R)$	$[2, -1]$ $[1, -2]$	Q Q	.



electroweak family

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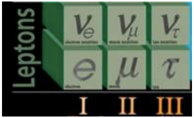
- $\mathcal{G}$ assumed to be a finite subgroup of $SU(3)$
- λ, ϵ, ρ: irreducible representations of $\mathcal{G}$



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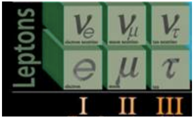
$e_a := (e_L, \mu_L, \tau_L)$	$[2, -1]$	Q	λ	$F \quad (F^n = 1)$
$E_a := (e_R, \mu_R, \tau_R)$	$[1, -2]$	Q	ϵ	F .

- $\mathcal{G}$ assumed to be a finite subgroup of $SU(3)$
- λ, ϵ, ρ: irreducible representations of $\mathcal{G}$
- $F, G \in \mathcal{G}$ generates $Z_n, Z_2 \subset \mathcal{G}$ respectively
- assume $F^\dagger F = 1 = G^\dagger G$
- $F \neq G$, or else there would be no neutrino mixing



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- $F \neq G$, or else there would be no neutrino mixing
- $\{\mathcal{G}, F, G\} \leftrightarrow$ PMNS mixing matrix U (see Part II)

Parametrization of U

$$U = \begin{bmatrix} \downarrow & \downarrow & \\ cc' & cs' & se^{-i\delta} \\ -s'c'' - sc's''e^{i\delta} & +c'c'' - ss's''e^{i\delta} & cs'' \leftarrow \\ +s's'' - sc'c''e^{i\delta} & -c's'' - ss'c''e^{i\delta} & cc'' \leftarrow \end{bmatrix}$$

$$s = s_{13},\ s' = s_{12},\ s'' = s_{23}$$

reactor solar atmospheric

Results

Before T2K

- $\theta_{12} = 34^\circ,\ \theta_{13} = 0,\ \theta_{23} = 45^\circ$
- $U \simeq$ tri-bimaximal
- $\{\mathcal{G}, F, G\} = \{S_4, Z_3, Z_2\}$
- $F^3 = 1,\ \ G^2 = 1$

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After T2K

- $\theta_{12} = 34^\circ,\ \theta_{13} = 9^\circ,\ \theta_{23} < 45^\circ$
- after scanning through all the nontrivial subgroups of $SU(3)$ of order <512 for $\mathcal{G}$ (59 of them) (arXiv: 1208.5527) $\Rightarrow$
- without tunable parameters, only

  $\{\mathcal{G}, F, G\} = \boxed{\{\Delta(150), Z_3, Z_2\}}$

  works (see Part III). It predicts

Group Theory (zero parameter prediction)

- $\sin^2 2\theta_{13} = 0.11$
- $\sin^2 2\theta_{23} = 0.94$

(no restriction on θ_{12}, m_i, δ)

Experiment (Kyoto Conf., June 2012)

- Daya Bay:

  $\sin^2 2\theta_{13} = 0.089 \pm 0.010 \pm 0.005$
- Double Chooz:

  $\sin^2 2\theta_{13} = 0.109 \pm 0.030 \pm 0.025$
- RENO:

  $\sin^2 2\theta_{13} = 0.113 \pm 0.013 \pm 0.019$
- T2K: (N)/(I, Prelim)

  $\sin^2 2\theta_{13} = 0.104 + 0.060 - 0.045$

  $\sin^2 2\theta_{13} = 0.128 + 0.070 - 0.055$
- MINOS: $(\nu)/(\overline{\nu})$

  $\sin^2 2\theta_{23} = 0.96 \pm 0.04$

  $\sin^2 2\theta_{23} = 0.97 \pm 0.03/0.08$

Dynamical Theory (tree order)

(additional zero parameter predictions)

- $\sin^2 2\theta_{12} = 0.90$
- $\delta = 0$
- $m_2 = 0$
- higher order correction is called for

Experiment (Kyoto Conf., June 2012)

- PDG:

  $\sin^2 2\theta_{12} = 0.95 \pm 0.10 \pm 0.01$
- $\delta =?$
- $(\Delta m)^2_{23} = 2.32 \times 10^{-3}\mathrm{eV}^2$

  $(\Delta m)^2_{12} = 7.59 \times 10^{-5}\ \mathrm{eV}^2$

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  $(\Delta m)^2_{12} = 7.59 \times 10^{-5}\ \mathrm{eV}^2$

$\Delta(150)$ gives rise to a reasonable theory without any tunable parameter!

II

Symmetry and Mixing

(Group Theory)

$$-\mathcal{L} = \overline{E}_a M^e_{ab} e_b + N_a M^\nu_{ab} \nu_b + \frac{1}{2} N_a M^N_{ab} N_b + \mathrm{h.c.} \qquad (M_{ab} = Y_{ab}\langle H \rangle)$$

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effective LL mass matrix

- $\overline{M}^e := M^{e\dagger} M^e = \overline{M}^{e\dagger}$
- $\overline{M}^\nu := M^{\nu\,T} \frac{1}{M^N} M^\nu = \overline{M}^{\nu\,T}$ (type-I seesaw)

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Neutrino Mixing Matrix U

- $U^T \overline{M}^\nu U$ is diagonal
  when $\overline{M}^e$ is diagonal
- $U = \begin{bmatrix} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

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residual family symmetry

- residual symmetry: $\mathcal{L} \to \mathcal{L}$ when

$$e \to F_\lambda e, \quad E \to F_\epsilon E,$$
$$\nu \to G_\lambda \nu, \quad N \to G_\rho N$$

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- $M^e = F^\dagger_\epsilon M^e F_\lambda$

  $\overline{M}^e = F^\dagger_\lambda \overline{M}^e F_\lambda$

  $M^\nu = G^\dagger_\rho M^\nu G_\lambda, \; M^N = G^T_\rho M^N G_\rho$

  $\overline{M}^\nu = G^T_\lambda \overline{M}^\nu G_\lambda$

$$-\mathcal{L} = \overline{E}_a M^e_{ab} e_b + N_a M^\nu_{ab} \nu_b + \frac{1}{2} N_a M^N_{ab} N_b + \mathrm{h.c.} \qquad (M_{ab} = Y_{ab}\langle H \rangle)$$

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- $M^e = F_\epsilon^\dagger M^e F_\lambda$

  $\overline{M}^e = F_\lambda^\dagger \overline{M}^e F_\lambda$

  $M^\nu = G_\rho^\dagger M^\nu G_\lambda, \; M^N = G_\rho^T M^N G_\rho$

  $\overline{M}^\nu = G_\lambda^T \overline{M}^\nu G_\lambda$

- $G_\lambda = u_i u_i^\dagger - u_j u_j^\dagger - u_k u_k^\dagger \quad \Rightarrow$

  $G_\lambda^2 = 1, \quad G_\lambda u_i = u_i$

  (i,j,k are permutations of 1,2,3)

$\mathcal{G}$ to U

- given $\mathcal{G}$, choose a nondegenerate $F \in \mathcal{G}$ of order $n > 2$, and a $G \in \mathcal{G}$ of order 2
- non-degeneracy of F ensures $\overline{M}_e$ to be diagonal when F is
- choose any 3-dimensional unitary irreducible representation
- compute the invariant eigenvector u of G in the F-diagonal representation
- u constitutes a column of the mixing matrix U. Compare with experiment

III

$\Delta(150)$

Structure and Representation

structure

- $\Delta(150) = (Z_5 \times Z_5) \rtimes Z_3 \rtimes Z_2$

  $f_4 \quad f_3 \quad f_2 \quad f_1$

- $f_4^d \; f_3^c \; f_2^b \; f_1^a \in \mathcal{G}$

- $f_2 f_4 = f_4 f_3 f_2, \quad f_2 f_3 = f_4^2 f_3^3 f_2$

  $f_1 f_4 = f_4^4 f_3^4 f_1, \quad f_1 f_3 = f_3 f_1,$

  $f_1 f_2 = f_2^2 f_1$

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- 13 classes, of orders
  1(1), 2(1), 3(1), 5(6), 10(4)

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representation

- 13 irreducible representations, of dimensions 1(2), 2(1), 3(8), 6(2)

Structure and Representation

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- 13 classes, of orders 1(1), 2(1), 3(1), 5(6), 10(4)

representation

- 13 irreducible representations, of dimensions 1(2), 2(1), 3(8), 6(2)
- the 3-dim and 6-dim representations are complex

  {representation of e and ν: $\lambda = 5$}

- the 1-dim and 2-dim representations are real

  {representation of N, E: $\rho, \epsilon = [3, 1]$ (or $[3, 2]$)}

Group Theory of Mixing

$$F_\lambda^\dagger \overline{M}^e F_\lambda = \overline{M}^e = \overline{M}^{e\dagger} \qquad G_\lambda^T \overline{M}^\nu G_\lambda = \overline{M}^\nu = \overline{M}^{\nu\,T}$$

charged leptons $(\omega = e^{2\pi i/3},\ \eta = e^{2\pi i/5})$ neutrino

- $F_\lambda = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
- $\overline{M}^e = \begin{bmatrix} \alpha & \beta & \beta^* \\ \beta^* & \alpha & \beta \\ \beta & \beta^* & \alpha \end{bmatrix}$

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- $\overline{M}^e = \begin{bmatrix} \alpha & \beta & \beta^* \\ \beta^* & \alpha & \beta \\ \beta & \beta^* & \alpha \end{bmatrix}$
- $V = \frac{1}{\sqrt{3}} \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \\ 1 & 1 & 1 \end{bmatrix}$
- $V^\dagger F V = \mathrm{diag}(\omega^2, \omega, 1)$
- $V^\dagger \overline{M}^e V = \mathrm{diag}$

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- $V^\dagger F V = \mathrm{diag}(\omega^2, \omega, 1)$
- $V^\dagger \overline{M}^e V = \mathrm{diag}$

- $G_\lambda = -\begin{bmatrix} 0 & \eta^3 & 0 \\ \eta^2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\overline{M}^\nu = \begin{bmatrix} a & b & c \\ b & a\eta & c\eta^3 \\ c & c\eta^3 & f \end{bmatrix}$

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- $V = \frac{1}{\sqrt{3}} \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \\ 1 & 1 & 1 \end{bmatrix}$
- $V^\dagger F V = \mathrm{diag}(\omega^2, \omega, 1)$
- $V^\dagger \overline{M}^e V = \mathrm{diag}$

- $G_\lambda = -\begin{bmatrix} 0 & \eta^3 & 0 \\ \eta^2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\overline{M}^\nu = \begin{bmatrix} a & b & c \\ b & a\eta & c\eta^3 \\ c & c\eta^3 & f \end{bmatrix}$
- $u_3' := \frac{1}{\sqrt{2}} \begin{bmatrix} -\eta^3 \\ 1 \\ 0 \end{bmatrix}$, $\quad G_\lambda u_3' = u_3'$, $\ \overline{M}^\nu u_3' = m_3 u_3'^*$, $\ m_3 = a\eta - b\eta^3$
- $|u_3| = |V^\dagger u_3'| = \begin{bmatrix} .179 \\ .607 \\ .777 \end{bmatrix} \Rightarrow$ $\sin^2 2\theta_{13} = 0.11$, $\sin^2 2\theta_{23} = 0.94$

$$-\mathcal{L} = \overline{E}_c M^e_{ca} e_a + N_c M^\nu_{ca} \nu_a + \tfrac{1}{2} N_c M^N_{ca} N_a + \text{h.c.}$$

residual symmetry : $\mathcal{L} \to \mathcal{L}$
$e \to F_\lambda e, \quad E \to F_\epsilon E$
$\nu \to G_\lambda \nu, \quad N \to G_\rho N$

General

- $\mathcal{G}$-invariant $\mathcal{L}$

$$M_{ca} \to \sum_b h_B \langle Cc|Bb, Aa\rangle \phi^B_b$$

$$-\mathcal{L} = \overline{E}_c M^e_{ca} e_a + N_c M^\nu_{ca} \nu_a + \frac{1}{2} N_c M^N_{ca} N_a + \text{h.c.}$$

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- $\mathcal{G} \to (F, G)$ (residual symmetry)

$$\phi^B \to \langle \phi^B \rangle$$

$$B \langle \phi^B \rangle = \langle \phi^B \rangle$$

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e : $C = F_{[3,1]}, \ A = F_5$

ν : $C = G_{[3,1]}, \ A = G_5$

N: $C = G_{[3,1]}, \ A = G_{[3,1]}$

$$-\mathcal{L} = \overline{E}_c M^e_{ca} e_a + N_c M^\nu_{ca} \nu_a + \frac{1}{2} N_c M^N_{ca} N_a + \text{h.c.}$$

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Charged Leptons

- $$F_{[3,1]} = \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- $$M^e = \begin{bmatrix} x\omega & x\omega^2 & x \\ y\omega^2 & y\omega & y \\ z & z & z \end{bmatrix}$$

$$-\mathcal{L} = \overline{E}_c M^e_{ca} e_a + N_c M^\nu_{ca} \nu_a + \frac{1}{2} N_c M^N_{ca} N_a + \text{h.c.}$$

residual symmetry $:\mathcal{L} \to \mathcal{L}$

$e \to F_\lambda e, \quad E \to F_\epsilon E$

$\nu \to G_\lambda \nu, \quad N \to G_\rho N$

General

- $\mathcal{G}$-invariant $\mathcal{L}$

$$M_{ca} \to \sum_b h_B \langle Cc | Bb, Aa \rangle \phi^B_b$$

- $\mathcal{G} \to (F, G)$ (residual symmetry)

$\phi^B \to \langle \phi^B \rangle$

$B\langle \phi^B \rangle = \langle \phi^B \rangle$

e : $C = F_{[3,1]}, \; A = F_5$

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- $V^\dagger F_5 V = \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- $V^\dagger \overline{M}^e V = 3 \begin{bmatrix} |y|^2 & 0 & 0 \\ 0 & |x|^2 & 0 \\ 0 & 0 & |z|^2 \end{bmatrix}$

Neutrino (Group Theory) (Dynamical Theory) $\nu \sim 5, \; N \sim [3,1]$

- $$M^{\nu} = \begin{bmatrix} \alpha & \beta & \gamma \\ -\beta\eta^2 & -\alpha\eta^3 & -\gamma \\ \delta & \delta\eta^3 & \epsilon \end{bmatrix}$$

- $$(M^N)^{-1} = \begin{bmatrix} A & B & C\eta^2 \\ B & A\eta & C \\ C\eta^2 & C & D \end{bmatrix}$$

- $$\overline{M}^{\nu} = \begin{bmatrix} a & b & c \\ b & a\eta & c\eta^3 \\ c & c\eta^3 & f \end{bmatrix}$$

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- $|u_3| = |V^\dagger u_3'| = \begin{bmatrix} .179 \\ .607 \\ .777 \end{bmatrix}$

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- $u_1' := \frac{1}{\sqrt{2}} \begin{bmatrix} \eta^3 \\ 1 \\ 0 \end{bmatrix}$, $\quad u_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

  $m_1 = a\eta + b\eta^3, \; m_2 = 0$

- $|u_1| = |V^\dagger u_1'| = \begin{bmatrix} .799 \\ .547 \\ .252 \end{bmatrix}$

  $|u_2| = |V^\dagger u_2'| = \begin{bmatrix} .577 \\ .577 \\ .577 \end{bmatrix}$

Solar Angle and CP Phase

$$|U| = \begin{bmatrix} .799 & .577 & .179 \\ .547 & .577 & .607 \\ .252 & .577 & .777 \end{bmatrix}$$

$$U = \begin{bmatrix} \downarrow & \downarrow & \\ cc' & cs' & se^{-i\delta} \\ -s'c'' - sc's''e^{i\delta} & +c'c'' - ss's''e^{i\delta} & cs'' \leftarrow \\ +s's'' - sc'c''e^{i\delta} & -c's'' - ss'c''e^{i\delta} & cc'' \leftarrow \end{bmatrix}$$

$$s = s_{13}, \;\; s' = s_{12}, \;\; s'' = s_{23}$$

$\sin^2 2\theta_{13} = 0.11,$ $\sin^2 2\theta_{23} = 0.94,$ $\sin^2 2\theta_{12} = 0.90,$ $\delta = 0$

Conclusion $\Delta(150)$

		Theory	Experiment
Group Theory	$\sin^2 2\theta_{12}$	0.11	$.089 \pm .010 \pm .005 \mid .109 \pm .030 \pm .025$ $.113 \pm .013 \pm .019 \mid .104 \pm .060/.045$
	$\sin^2 2\theta_{23}$	0.94	$.96 \pm .04 \mid .97 \pm .03/.08$
Dynamical Theory (tree)	$\sin^2 2\theta_{12}$	0.90	$.95 \pm .10 \pm .01$
	δ	0	?
	m_2	0	$(\Delta m)^2_{23} = 2.32 \times 10^{-3} \mathrm{eV}^2$ $(\Delta m)^2_{12} = 7.59 \times 10^{-5}\ \mathrm{eV}^2$

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- Theoretical predictions contain no adjustable parameters
- Mixing Parameters: good approximation to reality
- Neutrino Masses: $m_2 = 0 \ \Rightarrow m_1 = 0$, fair approximation as

$$(\Delta m_{12})^2 \ll (\Delta m_{23})^2$$

- Tree theory gives a reasonable prediction, though a small higher order correction is needed for accuracy