Neutrinos in core-collapse supernovae: symmetries

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Neutrinos and New Physics
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In memory of Stuart Freedman (1944-2012)

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Motivation

Supernova neutrinos

- $M_{\mathrm{progenitor}} \geq 8 M_{\odot} \Rightarrow \Delta E \sim 10^{59}$ MeV
- 99 % of this energy is carried away by neutrinos and antineutrinos with $10 \leq E_{\nu} \leq 30$ MeV $\Rightarrow 10^{58}$ neutrinos!
  A neutrino many-body system!

Neutrino many-body system

Many neutrino system

This is the only many-body system driven by the weak interactions:

Table: Many-body systems

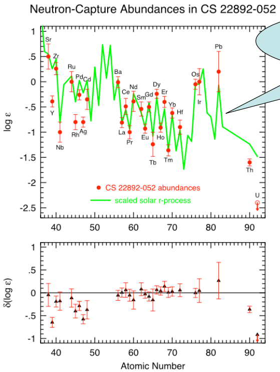
Nuclei	Strong	at most ∼250 particles
Condensed matter	E&M	at most N_A particles
ν's in SN	Weak	$\sim 10^{58}$particles

Astrophysical extremes allow us to test physics that cannot be tested elsewhere!

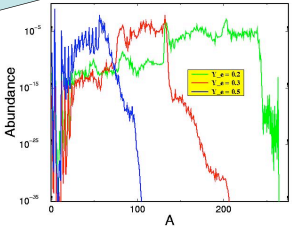
r-process Nucleosynthesis

[Fe/H] ≈ -3.1

r-process abundances



A > 100 abundance pattern fits the solar abundances well



r-process abundances should depend very strongly on electron fraction Meyer

r-process Nucleosynthesis

- Yields of r-process nucleosynthesis are determined by the electron fraction, or equivalently by the neutron-to-proton ratio, n/p
- Interactions of the neutrinos and antineutrinos streaming out of the core both with nucleons and seed nuclei determine the n/p ratio. Hence it is crucial to understand neutrino properties and interactions.
- As these neutrinos reach the r-process region they undergo matter-enhanced neutrino oscillations as well as coherently scatter over other neutrinos. Many-body behavior of this neutrino gas is being explored, but may have significant impact on r-process nucleosynthesis.

Neutrino Mixing

Mass and Flavor States

$$a_1(\mathbf{p}, s) = \cos\theta \, a_e(\mathbf{p}, s) - \sin\theta \, a_x(\mathbf{p}, s)$$

$$a_2(\mathbf{p}, s) = \sin\theta \, a_e(\mathbf{p}, s) + \cos\theta \, a_x(\mathbf{p}, s)$$

Flavor Isospin Operators

$$\hat{J}^+_{\mathbf{p},s} = a_e^\dagger(\mathbf{p}, s) a_x(\mathbf{p}, s) \ , \qquad \hat{J}^-_{\mathbf{p},s} = a_x^\dagger(\mathbf{p}, s) a_e(\mathbf{p}, s) \ ,$$

$$\hat{J}^0_{\mathbf{p},s} = \frac{1}{2}\left(a_e^\dagger(\mathbf{p}, s) a_e(\mathbf{p}, s) - a_x^\dagger(\mathbf{p}, s) a_x(\mathbf{p}, s) \right)$$

$$[\hat{J}^+_{\mathbf{p},s}, \hat{J}^-_{\mathbf{q},r}] = 2\delta_{\mathbf{pq}}\delta_{sr}\hat{J}^0_{\mathbf{p},s} \ , \qquad [\hat{J}^0_{\mathbf{p},s}, \hat{J}^\pm_{\mathbf{q},r}] = \pm\delta_{\mathbf{pq}}\delta_{sr}\hat{J}^\pm_{\mathbf{p},s} \ ,$$

Neutrino Hamiltonian

Vacuum Oscillation Term

$$\hat{H}_\nu^{(1)} = \sum_{\mathbf{p},s} \left(\frac{m_1^2}{2p} a_1^\dagger(\mathbf{p},s) a_1(\mathbf{p},s) + \frac{m_2^2}{2p} a_2^\dagger(\mathbf{p},s) a_2(\mathbf{p},s) \right) .$$

$$\hat{H}_\nu^{(1)} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p$$

$$\hat{B} = (\sin 2\theta, 0, -\cos 2\theta)$$

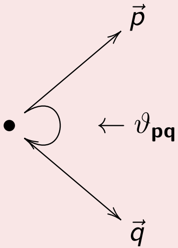
One-Body Hamiltonian including interactions with the electron background

$$\hat{H}_\nu = \sum_p \left(\frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sqrt{2} G_F N_e J_p^0 \right)$$

Neutrino Hamiltonian

Neutrino-Neutrino Interactions

$$\hat{H}_{\nu\nu} = \frac{\sqrt{2}G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}})\, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$



$(1 - cos\vartheta)$ terms follow from the V-A nature of the weak interactions.

Neutrino Hamiltonian

The total neutrino Hamiltonian

$$\hat{H}_{\text{total}} = H_\nu + H_{\nu\nu} = \left(\sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sqrt{2} G_F N_e J_p^0 \right) + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

Pantaleone, Dasgupta, Duan, Fogli, Fuller, Kostelecky, McKellar, Lisi, Mirizzi, Qian, Pastor, Raffelt, Samuel, Sawyer, Sigl, Smirnov, . . .

Evolution Equations

Path Integral for the Evolution Operator

$$i\frac{\partial U}{\partial t} = (H_\nu + H_{\nu\nu})\, U$$

Use SU(2) coherent states to write the evolution operator as a path integral:

$$|z(t)\rangle = \exp\left(\int dp z(p,t) J_+(p)\right)|\phi\rangle$$

$$|\phi\rangle = \prod_p a_e^\dagger(p)|0\rangle$$

$$\langle z'(t_f)|U|z(t_i)\rangle = \int \mathcal{D}[z,z^*]\exp\left(iS[z,z^*]\right)$$

Calculating the Evolution Operator

Stationary Phase Approximation

$$\langle z'(t_f)|U|z(t_i)\rangle = \int \mathcal{D}[z,z^*] \exp\left(iS[z,z^*]\right)$$

$$S(z,z^*) = \int_{t_i}^{t_f} dt \frac{\langle z(t)|i\frac{\partial}{\partial t} - H(t)|z(t)\rangle}{\langle z(t)|z(t)\rangle} + \log\langle z'(t_f)|z(t_f)\rangle$$

$$H = H_\nu + H_{\nu\nu}$$

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{z}} - \frac{\partial}{\partial z}\right) L(z,z^*) = 0 \qquad \left(\frac{d}{dt}\frac{\partial}{\partial \dot{z}^*} - \frac{\partial}{\partial z^*}\right) L(z,z^*) = 0$$

$$i\dot{z}(p,t) = \beta(p,t) - \alpha(p,t)z(p,t) - \beta^*(p,t)z(p,t)^2$$

$$\alpha(p,t) = -\frac{\delta m^2}{2p}\cos 2\theta + \sqrt{2}G_F N_e + \sqrt{2}G_F \int dq(1-\cos\theta_{pq})\left(\frac{1-|z(q,t)|^2}{1+|z(q,t)|^2}\right)$$

$$\beta(p,t) = \frac{1}{2}\frac{\delta m^2}{2p}\sin 2\theta + \sqrt{2}G_F \int dq(1-\cos\theta_{pq})\left(\frac{z(q,t)}{1+|z(q,t)|^2}\right)$$

$$z(p,t) = \frac{\psi_x(p,t)}{\psi_e(p,t)}, \quad |\psi_e|^2 + |\psi_x|^2 = 1$$

The mean-field/RPA solution

$$z(p,t) = \frac{\psi_x(p,t)}{\psi_e(p,t)}, \quad |\psi_e|^2 + |\psi_x|^2 = 1$$

$$\Delta = \frac{\delta m^2}{2p}, \qquad A = \sqrt{2} G_F N_e$$

$$D = \sqrt{2} G_F \int dq (1 - \cos\theta_{pq}) \left[\left(|\psi_e(q,t)|^2 - |\psi_x(q,t)|^2 \right) \right]$$

$$D_{ex} = 2\sqrt{2} G_F \int dq (1 - \cos\theta_{pq}) \left(\psi_e(q,t) \psi_x^*(q,t) \right)$$

$$i \frac{\partial}{\partial t} \begin{pmatrix} \psi_e \\ \psi_x \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A + D - \Delta \cos 2\theta & D_{e\mu} + \Delta \sin 2\theta \\ D_{\mu e} + \Delta \sin 2\theta & -A - D + \Delta \cos 2\theta \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_x \end{pmatrix}$$

This is the mean field approximation! Recall that one can approximate product of two commuting arbitrary operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ as

$$\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \sim \hat{\mathcal{O}}_1 \langle \xi | \hat{\mathcal{O}}_2 | \xi \rangle + \langle \xi | \hat{\mathcal{O}}_1 | \xi \rangle \hat{\mathcal{O}}_2 - \langle \xi | \hat{\mathcal{O}}_1 | \xi \rangle \langle \xi | \hat{\mathcal{O}}_2 | \xi \rangle,$$

provided that

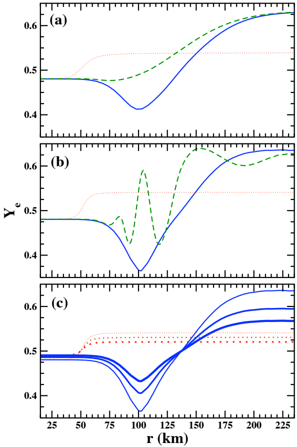
$$\langle \xi | \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 | \xi \rangle = \langle \xi | \hat{\mathcal{O}}_1 | \xi \rangle \langle \xi | \hat{\mathcal{O}}_2 | \xi \rangle.$$

This reduces $H_{\nu\nu}$ to a one-body Hamiltonian:

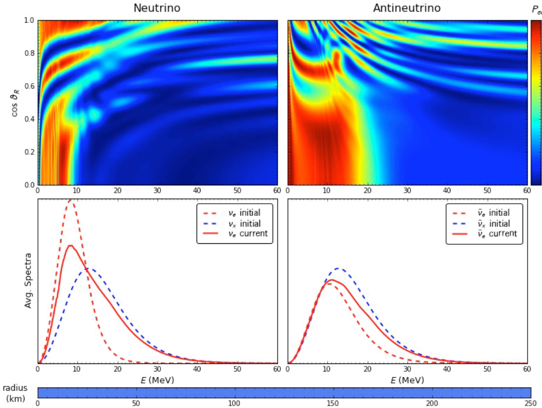
$$\begin{aligned} \mathcal{H}_{\nu\nu} \quad \sim \quad & 2 \frac{\sqrt{2} G_F}{V} \int d^3p \, d^3q \, R_{pq} \, \left(J_0(p) \langle J_0(q) \rangle \right. \\ + \quad & \left. \frac{1}{2} J_+(p) \langle J_-(q) \rangle + \frac{1}{2} J_-(p) \langle J_+(q) \rangle \right) \end{aligned}$$

Examples of mean-field calculations

- With ν luminosity $L^{51} =$ 0.001 (blue), 0.1 (green), 50 (red)
- Balantekin and Yüksel, New J. Phys. **7** 51 (2005).



Examples of mean-field calculations



Fuller, Qian, Carlson, Duan,...

Beyond the Mean field

Corrections to RPA

$$\beta(p,t) = \frac{z(p,t)}{\sqrt{1+|z(p,t)|^2}} \qquad \beta^*(p,t) = \frac{z^*(p,t)}{\sqrt{1+|z(p,t)|^2}}$$

$$\langle z'(t_f)|U|z(t_i)\rangle = \int \lim_{N\to\infty} \prod_{\alpha=1}^{N} \prod_{p\in\mathcal{P}} \frac{d\beta(p,t_\alpha)d\beta^*(p,t_\alpha)}{i\pi} e^{iS[\beta,\beta^*]}$$

$$\langle z'(t_f)|U|z(t_i)\rangle = \lim_{N\to\infty} (i\pi)^{N+P} \frac{e^{iS[\beta_{cl},\beta^*_{cl}]}}{\sqrt{Det\left(KM - L^T K^{-1} L\right)}}$$

$P =$ the number of allowed momentum modes.

$$K(p,k,q,m) = \frac{1}{2}\left(\frac{\delta^2 S}{\delta x(p,t_k)\,\delta x(q,t_m)}\right)_{cl}$$

$$M(p,k,q,m) = \frac{1}{2}\left(\frac{\delta^2 S}{\delta y(p,t_k)\,\delta y(q,t_m)}\right)_{cl}$$

$$L(p,k,q,m) = \frac{1}{2}\left(\frac{\delta^2 S}{\delta x(p,t_k)\,\delta y(q,t_m)}\right)_{cl}$$

$x = (\tilde\beta + \tilde\beta^*)/2,\ y = (\tilde\beta - \tilde\beta^*)/2i$
$\tilde\beta = \beta - \beta_{cl}$

Antineutrinos and three flavors

Including antineutrinos

$$H = H_\nu + H_{\bar{\nu}} + H_{\nu\nu} + H_{\bar{\nu}\bar{\nu}} + H_{\nu\bar{\nu}}$$

Requires introduction of a second set of SU(2) algebras!

Including three flavors

Requires introduction of SU(3) algebras.

Both extensions are straightforward, but tedious!
Balantekin and Pehlivan, J. Phys. G **34**, 1783 (2007).

Neutrino Hamiltonian

Neutrino Hamiltonian with $\nu - \nu$ interactions

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

Single-angle approximation $\Rightarrow$

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \vec{J} \cdot \vec{J}$$

Defining $\mu = \frac{\sqrt{2} G_F}{V}$, $\tau = \mu t$, and $\omega_p = \frac{1}{\mu} \frac{\delta m^2}{2p}$ one can write

$$\hat{H} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{J} \cdot \vec{J}$$

Conserved Quantities

Some Invariants

$$\hat{H} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{J} \cdot \vec{J}$$

This Hamiltonian preserves the *length of each spin*

$$\hat{L}_p = \vec{J}_p \cdot \vec{J}_p \,, \qquad \left[\hat{H}, \hat{L}_p\right] = 0 \,,$$

as well as the *total spin component* in the direction of the "external magnetic field", $\hat{B}$

$$\hat{C}_0 = \hat{B} \cdot \vec{J} \,, \qquad \left[\hat{H}, \hat{C}_0\right] = 0$$

Raffelt, Smirnov, Fuller, Pehlivan, Balantekin, Kajino, Yoshida, $\cdots$

BCS Hamiltonian

Hamiltonian in Quasi-spin basis

$$\hat{H}_{\text{BCS}} = \sum_k 2\epsilon_k \hat{t}^0_k - |G| \hat{T}^+ \hat{T}.$$

Quasi-spin operators:

$$\hat{t}^+_k = c^\dagger_{k\uparrow} c^\dagger_{k\downarrow}, \qquad \hat{t}^-_k = c_{k\downarrow} c_{k\uparrow}, \qquad \hat{t}^0_k = \frac{1}{2}\left(c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow} - 1\right)$$

$$[\hat{t}^+_k, \hat{t}^-_l] = 2\delta_{kl}\hat{t}^0_k \ , \qquad [\hat{t}^0_k, \hat{t}^\pm_l] = \pm\delta_{kl}\hat{t}^\pm_k \ .$$

Richardson gave a solution of this problem. Hence there exist invariants of motion.

The duality between $\nu - \nu$ and BCS Hamiltonians

The ν-ν Hamiltonian

$$\hat{H} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{J} \cdot \vec{J}$$

$\Longleftrightarrow$

The BCS Hamiltonian

$$\hat{H}_{\text{BCS}} = \sum_k 2\epsilon_k \hat{t}_k^0 - |G| \hat{T}^+ \hat{T}$$

Same symmetries leading to Analogous (dual) dynamics!
Pehlivan, Balantekin, Kajino, and Yoshida, Phys.Rev. D **84**, 065008 (2011).

Invariants

Invariants

The collective neutrino Hamiltonian given has the following constants of motion:

$$\hat{h}_p = \hat{B} \cdot \vec{J}_p + 2 \sum_{q(\neq p)} \frac{\vec{J}_p \cdot \vec{J}_q}{\omega_p - \omega_q}.$$

The individual neutrino spin-length discussed before in an independent invariant. However $\hat{C}_0 = \sum_p \hat{h}_p$. The Hamiltonian itself is also a linear combination of these invariants.

$$\hat{H} = \sum_p w_p \hat{h}_p + \sum_p \hat{L}_p \ .$$

Pehlivan, Balantekin, Kajino, Yoshida

Eigenvalues and Eigenstates

Eigenstates of the system

- $J_{\text{max}} = N/2$ N, the total number of neutrinos
- A state with all electron neutrinos:
  $|\nu_e\ \nu_e\ \nu_e\ \ldots\rangle = |J_{\text{max}}\ \ J_{\text{max}}\rangle_f$
- Matter and flavor bases are connected with a unitary transformation: $|J_{\text{max}}\ \ J_{\text{max}}\rangle_f = \hat{U}^\dagger |J_{\text{max}}\ \ J_{\text{max}}\rangle_m$
- $|J_{\text{max}}\ \ J_{\text{max}}\rangle_m = \prod_{\mathbf{p},s} a_1^\dagger(\mathbf{p},s)\ |0\rangle$
  $|J_{\text{max}}\ \ -J_{\text{max}}\rangle_m = \prod_{\mathbf{p},s} a_2^\dagger(\mathbf{p},s)\ |0\rangle$
  $E_{(+J_{\text{max}})} = -\sum_p \frac{n_p}{2}\omega_p + J_{\text{max}}\left(J_{\text{max}}+1\right)$
  $E_{(-J_{\text{max}})} = \sum_p \frac{n_p}{2}\omega_p + J_{\text{max}}\left(J_{\text{max}}+1\right)$

Eigenvalues and Eigenstates

Other states

$$\mathcal{Q}^{\pm}(\xi) == \sum_p \frac{1}{\omega_p - \xi}\left(\cos^2\theta \hat{J}_p^{\pm} + \sin 2\theta \hat{J}_p^0 - \sin^2\theta \hat{J}_p^{\mp}\right)$$

$$\begin{aligned}\hat{H}\mathcal{Q}^{+}(\xi)|J\ -J\rangle_m &= \left(E_{(-J)} - 2J - \xi\right)\mathcal{Q}^{+}(\xi)|J\ -J\rangle_m \\ &+ \underbrace{\left(1 + 2\sum_p \frac{-j_p}{w_p - \xi}\right)\mathcal{Q}^{+}|J\ -J\rangle_m}_{\text{should be zero if eigenstate}}\end{aligned}$$

This gives us the Bethe ansatz equation $\Rightarrow \sum_p \frac{-j_p}{w_p - \xi} = -\frac{1}{2}$

Eigenvalues and Eigenstates

Most General Eigenstate

$$|\xi_1, \xi_2, \dots \xi_\kappa\rangle \equiv \mathcal{Q}^+(\xi_1)\mathcal{Q}^+(\xi_2)\dots\mathcal{Q}^+(\xi_\kappa)|J \quad -J\rangle_m$$

$$E(\xi_1, \xi_2, \dots, \xi_\kappa) = E_{(-J)} - \sum_{\alpha=1}^{\kappa} \xi_\alpha - \kappa(2J - \kappa + 1) ,$$

$$\underbrace{\sum_p \frac{-j_p}{\omega_p - \xi_\alpha} = -\frac{1}{2} + \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{\kappa} \frac{1}{\xi_\alpha - \xi_\beta} .}_{\text{Bethe ansatz equations}}$$

An RPA-like approximation

An RPA-inspired approximation when $[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] = 0$. Approximate the operator product as

$$\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2 \sim \hat{\mathcal{O}}_1\langle\hat{\mathcal{O}}_2\rangle + \langle\hat{\mathcal{O}}_1\rangle\hat{\mathcal{O}}_2 - \langle\hat{\mathcal{O}}_1\rangle\langle\hat{\mathcal{O}}_2\rangle \ ,$$

where the expectation values should be calculated with respect to a state $|\Psi\rangle$ which satisfies the condition $\langle\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2\rangle = \langle\hat{\mathcal{O}}_1\rangle\langle\hat{\mathcal{O}}_2\rangle$.

$$\hat{H} \sim \hat{H}^{\text{RPA}} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J}$$

Polarization vector: $\vec{P}_{\mathbf{p},s} = 2\langle\vec{J}_{\mathbf{p},s}\rangle$. Use SU(2) coherent states for the expectation value.

Mean-neutrino field

Polarization vectors

$$\hat{H} \sim \hat{H}^{\mathrm{RPA}} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J}$$

$$\vec{P}_{\mathbf{p},s} = 2\langle \vec{J}_{\mathbf{p},s} \rangle$$

Eqs. of motion: $$\frac{d}{d\tau}\vec{J}_p = -i[\vec{J}_p, \hat{H}^{\mathrm{RPA}}] = (\omega_p \hat{B} + \vec{P}) \times \vec{J}_p$$

RPA Consistency requirement $\Rightarrow$ $$\frac{d}{d\tau}\vec{P}_p = (\omega_p \hat{B} + \vec{P}) \times \vec{P}_p$$

Invariants $$I_p = 2\langle \hat{h}_p \rangle = \hat{B} \cdot \vec{P}_p + \sum_{q(\neq p)} \frac{\vec{P}_p \cdot \vec{P}_q}{\omega_p - \omega_q} \Rightarrow \frac{d}{d\tau} I_p = 0$$

Raffelt; Pehlivan, Balantekin, Kajino, Yoshida

Total Hamiltonian

Hamiltonian with both ν's and $\bar{\nu}$'s

$$\begin{aligned}
\hat{H}_{\text{total}} &= \sum_p \frac{\delta m^2}{2p}\left(-\cos 2\theta\, \hat{J}^0_p + \sin 2\theta\, \frac{\hat{J}^+_p + \hat{J}^-_p}{2}\right) \\
&+ \sum_{\bar{p}} \frac{\delta m^2}{2\bar{p}}\left(\cos 2\theta\, \hat{\bar{J}}^0_{\bar{p}} + \sin 2\theta\, \frac{\hat{\bar{J}}^+_{\bar{p}} + \hat{\bar{J}}^-_{\bar{p}}}{2}\right) \\
&+ \frac{\sqrt{2}G_F}{V}\left(\sum_{\mathbf{p},\mathbf{q}}(1-\cos\vartheta_{\mathbf{pq}})\vec{J}_{\mathbf{p}}\cdot\vec{J}_{\mathbf{q}} + \sum_{\bar{\mathbf{p}},\bar{\mathbf{q}}}(1-\cos\vartheta_{\bar{\mathbf{p}}\bar{\mathbf{q}}})\vec{\bar{J}}_{\bar{\mathbf{p}}}\cdot\vec{\bar{J}}_{\bar{\mathbf{q}}}\right. \\
&+ \left.\sum_{\mathbf{p},\bar{\mathbf{q}}}(1-\cos\vartheta_{\mathbf{p}\bar{\mathbf{q}}})\left(2\hat{J}^0_{\mathbf{p}}\hat{\bar{J}}^0_{\bar{\mathbf{q}}} - \hat{J}^+_{\mathbf{p}}\hat{\bar{J}}^-_{\bar{\mathbf{q}}} - \hat{J}^-_{\mathbf{p}}\hat{\bar{J}}^+_{\bar{\mathbf{q}}}\right)\right) .
\end{aligned}$$

Including antineutrinos

Single angle approximation

$$H_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sum_{\bar{p}} \frac{\delta m^2}{2\bar{p}} \hat{B} \cdot \vec{\tilde{J}}_p + \frac{\sqrt{2} G_F}{V} \left(\vec{J} + \vec{\tilde{J}} \right) \cdot \left(\vec{J} + \vec{\tilde{J}} \right)$$

Defining $\omega_{\bar{p}} = -\frac{1}{\mu} \frac{\delta m^2}{2\bar{p}}$, one writes

$$H = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \sum_{\bar{p}} \omega_{\bar{p}} \hat{B} \cdot \vec{\tilde{J}}_p + \left(\vec{J} + \vec{\tilde{J}} \right) \cdot \left(\vec{J} + \vec{\tilde{J}} \right)$$

Invariants

Invariants

Conserved quantities for each neutrino energy mode p:

$$\hat{h}_p = \hat{B} \cdot \vec{J}_p + 2 \sum_{q(\neq p)} \frac{\vec{J}_p \cdot \vec{J}_q}{\omega_p - \omega_q} + 2 \sum_{\bar{q}} \frac{\vec{J}_p \cdot \vec{\tilde{J}}_{\bar{q}}}{\omega_p - \omega_{\bar{q}}}$$

Conserved quantity $\hat{h}_{\bar{p}}$ for each antineutrino energy mode:

$$\hat{h}_{\bar{p}} = \hat{B} \cdot \vec{\tilde{J}}_p + 2 \sum_{\bar{q}(\neq \bar{p})} \frac{\vec{\tilde{J}}_{\bar{p}} \cdot \vec{\tilde{J}}_{\bar{q}}}{\omega_{\bar{p}} - \omega_{\bar{q}}} + 2 \sum_{q} \frac{\vec{\tilde{J}}_{\bar{p}} \cdot \vec{J}_q}{\omega_{\bar{p}} - \omega_q} .$$

Invariants

Mean-field Invariants

$$I_p = 2\langle \hat{h}_p \rangle = \hat{B} \cdot \vec{P}_p + \sum_{q(\neq p)} \frac{\vec{P}_p \cdot \vec{P}_q}{\omega_p - \omega_q} + \sum_{\bar{q}} \frac{\vec{P}_p \cdot \vec{\tilde{P}}_{\bar{q}}}{\omega_p - \omega_{\bar{q}}}$$

$$I_{\bar{p}} = 2\langle \hat{h}_{\bar{p}} \rangle = \hat{B} \cdot \vec{\tilde{P}}_{\bar{p}} + \sum_{\bar{q}(\neq \bar{p})} \frac{\vec{\tilde{P}}_{\bar{p}} \cdot \vec{\tilde{P}}_{\bar{q}}}{\omega_{\bar{p}} - \omega_{\bar{q}}} + \sum_{q} \frac{\vec{\tilde{P}}_{\bar{p}} \cdot \vec{P}_q}{\omega_{\bar{p}} - \omega_q}$$

Raffelt; Pehlivan *et al.*

Spectral Splits/swaps

Lagrange multiplier to enforce neutrino number conservation:

$$\hat{H}^{\rm RPA} + \omega_c \hat{J}^0 = \sum_p (\omega_c - \omega_p)\hat{J}^0_p + \vec{\mathcal{P}} \cdot \vec{J}$$

$$= \sum_{\mathbf{p},s} 2\lambda_p \hat{U}'^{\dagger} \hat{J}^0_p \hat{U}'$$

$$\hat{U}' = e^{\sum_p z_p J_p^+} \, e^{\sum_p \ln(1+|z_p|^2) J_p^0} \, e^{-\sum_p z_p^* J_p^-}$$

$$z_p = e^{i\delta} \tan\theta_p$$

$$\cos\theta_p = \sqrt{\frac{1}{2}\left(1 + \frac{\omega_c - \omega_p + \mathcal{P}^0}{2\lambda_p}\right)}$$

Spectral Splits/swaps

Many people contributed to their explanation

Raffelt, Mirizzi, Dasgupta, Smirnov, Fuller, Qian, Duan, Carlson$\cdots$

$$\begin{aligned} \alpha_1(\mathbf{p},s) &= \hat{U}'^{\dagger} a_1(\mathbf{p},s)\hat{U}' = \cos\theta_p \; a_1(\mathbf{p},s) - e^{i\delta}\sin\theta_p \; a_2(\mathbf{p},s) \\ \alpha_2(\mathbf{p},s) &= \hat{U}'^{\dagger} a_2(\mathbf{p},s)\hat{U}' = e^{-i\delta}\sin\theta_p \; a_1(\mathbf{p},s) + \cos\theta_p \; a_2(\mathbf{p},s) \end{aligned}$$

$$\hat{H}^{\mathrm{RPA}} + \omega_c \hat{J}^0 = \sum_{\mathbf{p},s} \lambda_p \left(\alpha_1^{\dagger}(\mathbf{p},s)\alpha_1(\mathbf{p},s) - \alpha_2^{\dagger}(\mathbf{p},s)\alpha_2(\mathbf{p},s) \right)$$

Spectral Splits/swaps

Assume that initially ($\mu \to \infty$) there are more ν_e's and all neutrinos are in flavor eigenstates:

$$\cos\theta_p = \sqrt{\frac{1}{2}\left(1 + \frac{P^0}{|\vec{P}|}\cos 2\theta\right)} \to_{\lim \mu\to\infty} \cos\theta$$

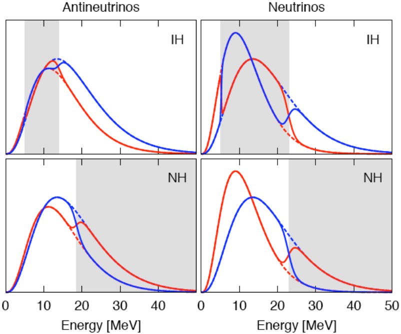
$$\alpha_1(\mathbf{p}, s) = \hat{U}^\dagger a_1(\mathbf{p}, s)\hat{U} \Rightarrow a_e(\mathbf{p}, s)$$

At the end ($\mu \to 0$)

$$\cos\theta_p = \sqrt{\frac{1}{2}\left(1 + \frac{\omega_c - \omega_p}{|\omega_c - \omega_p|}\right)} \Rightarrow \begin{cases} 1 & \omega_p < \omega_c \\ 0 & \omega_p > \omega_c \end{cases}$$

$$\alpha_1(\mathbf{p}, s) = \hat{U}^\dagger a_1(\mathbf{p}, s)\hat{U} \Rightarrow a_1(\mathbf{p}, s)$$

Spectral Splits/swaps



from Dasgupta *et al.*

Conclusions

Conclusions

- We examined the many-neutrino gas both from the exact many-body perspective and an effective one-body description formulated with the application of the RPA method. In the limit of the single angle approximation, both pictures possess many constants of motion manifesting the existence of associated dynamical symmetries in the system.
- The existence of constants of motion offer practical ways of extracting information even from exceedingly complex systems. Even when the symmetries which guarantee their existence is broken, they usually provide a convenient set of variables which behave in a relatively simple manner depending on how drastic the symmetry breaking factor is.

Conclusions

Conclusions - continued

- The existence of such invariants naturally lead to associated collective modes in neutrino oscillations. However, symmetries alone do not guarantee the stability of such collective behavior. An extensive numerical study of the collective neutrino phenomena associated with our invariants would shed light on the question of stability.