The Jeans Instability

The Gravitational Collapse of Hot Gas

Hot gases over a certain size are likely to undergo gravitational collapse. Consider a gas at a certain temperature. The thermal fluctuations of the gas is mediated by sound waves propagating at speed c_s. Density fluctuations of the gas will tend to induce gravitational collapse but they will tend to be smoothed out by thermal fluctuations. However, if the density fluctuation is on a length scale that is so large that sounds waves cannot prevent parts of the gas from free fall then the gas will undergo gravitational collapse. This is the essence of the Jeans instability. In what follows, we shall flesh this out in more detail.

The Equations of Fluid Dynamics

Consider a fluid with density ρ, pressure P and velocity $\mathbf{v}$ in a gravitational potential ϕ. Then the equation of motion of such a fluid is given by Euler's equation (which is just Newton's equation for the fluid):

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \phi. \tag{1}$$

In addition, we also have the equation of continuity and Poisson's equation for gravity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2}$$

$$\nabla^2 \phi = 4\pi G \rho, \tag{3}$$

where G is Newton's constant.

Linearized Equations

Consider a uniform and static ideal gas. Then consider the following perturbations

$$\begin{aligned}\rho &= \rho_0 + \rho_1 \\ P &= P_0 + P_1 \\ \mathbf{v} &= \mathbf{v}_1 \\ \phi &= \phi_0 + \phi_1\end{aligned}$$

where the background values have been denotes by the subscript 0 (unless it vanishes) and the small perturbations have been labelled by the subscript 1. If we assume that the perturbations are adiabatic then the density and pressure perturbations are related by the speed of sound:

$$c_s^2 = \frac{P_1}{\rho_1}.$$

- What are the linear versions of equations (1), (2) and (3)?
- Combine these equations to obtain a second order wave equation (with source) for the density fluctuation.
- Assume a plane wave ansatz for the density fluctuation. Show that the dispersion relation the follows from the wave-equation is

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0.$$

- Derive the Jeans length λ_J, i.e. the minimal length above which fluctuations become unstable.
- The Jeans mass is the mass of the matter that is contained in a sphere of radius $\frac{1}{2}\lambda_J$. Derive the Jeans mass.