Tutorial # 1

1. **Regulated and Renormalized Integrals**
Here's a toy model for regularization and renormalization.

   a) Evaluate
$$I_2(m^2) = \int_0^{\Lambda} dx\, x^3 \frac{1}{x^2 + m^2}$$

   b) Evaluate
$$I_4(m^2) = \int_0^{\Lambda} dx\, x^3 \left(\frac{1}{x^2 + m^2}\right)^2$$

   c) Define "renormalized" functions by
$$\begin{aligned} \widetilde{I}_2(m^2) &= I_2(m^2) + \delta_2 M^2 + \tilde{\delta}_2(m^2 - M^2) \\ \widetilde{I}_4(m^2) &= I_4(m^2) + \delta_4 \end{aligned}$$
and choose δ_2 and δ_4 such that $\widetilde{I}_2(M^2) = 0 = \widetilde{I}_4(M^2)$ at the special point $m^2 = M^2$ as well as $d\widetilde{I}_2/dm^2(M^2) = 0$. With these choices, find the expressions for $\widetilde{I}_2(m^2)$ and $\widetilde{I}_4(m^2)$ at general values of m^2 assuming that $\Lambda^2 \gg m^2,\, M^2$. Show that these are finite as $\Lambda \to \infty$, and look at what happens to them when m^2 becomes much larger than M^2

2. **Dark Matter Thermal Freeze-Out Estimate**
About 25% of the energy density in the Universe today consists of dark matter (DM). The most promising explanation is a stable new particle. Consider a theory with a Dirac-fermion DM candidate χ and a real "SM" scalar boson ϕ interacting via
$$-\mathscr{L} \supset g\phi\overline{\psi}\psi$$

   a) Estimate the cross section times relative velocity (σv) for $\psi\overline{\psi} \to \phi\phi$ in the limit where the initial ψ particles are non-relativistic and much heavier than ϕ. Use dimensional analysis.
Hint: for the relative velocity part, take a look at how it appears in the cross section formula.

   b) The rate for these scatterings becomes too small to maintain an equilibrium density of ψ particles when the temperature of the Universe falls below the mass of ψ. This leaves a remnant *relic density* of ψ particles that act as dark matter. The relationship between σv and the fraction of the total cosmological energy density today is approximately
$$\rho_\chi/\rho_{total} \;\sim\; (0.25) \times \left(\frac{3 \times 10^{-26}\text{cm}^3/\text{s}}{\sigma v}\right) .$$
For $g \sim 0.5$, estimate the value of the ψ mass that gives the observed DM fractional relic density.

3. **Higher-Dimensional Operators**
These can do interesting things!

a) Neutrino masses can arise in the SM if we add the higher-dimensional operator

$$-\mathscr{L} \supset y_N^2 \frac{(H\cdot L)(H\cdot L)}{M_N} + (h.c.)$$

where $H = (H^+, H^0)$ is the Higgs doublet, $L = (\nu_L, \ell_L)$ is written as a two-component fermion (with fermion indices contracted in the usual two-component way), and $A\cdot B = A_a B_b \epsilon^{ab}$ is a gauge-invariant contraction of $SU(2)_L$ indices. Show that this operator generates a neutrino mass after the Higgs develops a VEV. For $y_N \sim 1$, estimate the mass scale M_N that gives $m_\nu \sim 0.1\,\text{eV}$.

b) Proton decay can occur through the operator

$$-\mathscr{L} \supset \frac{1}{M^2}\epsilon_{ijk}(Q^i\cdot Q^j)(Q^k\cdot L)\ ,$$

where we have written everything in terms of two-component fermions, ϵ_{ijk} produces a gauge-invariant contraction of $SU(3)_c$ indices, and the $SU(2)_L$ and fermion indices are contracted as above. Find a channel for proton decay induced by this operator, and estimate its rate using dimensional analysis. Given the current limit on the proton lifetime of about 10^{33}yrs, how large must M be?
Hint: the scale for all hadronic stuff is about $m_p \sim$ GeV. *Also,* $\hbar c = 0.197$ GeV *fm,* 1 *yr* $\simeq \pi \times 10^7$ *s,* $c \simeq 3 \times 10^{23}$*fm/s.*