Notes #4: Extra Dimensions

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March 11, 2015

A third approach to the electroweak hierarchy problem is to postulate that the fundamental Planck scale is not much larger than the electroweak scale. In these scenarios, the strength of gravity is diluted in a way that makes it appear to be much weaker (to us) than it really is. In particular, the true scale of quantum gravity is $M_* \ll M_{\rm Pl}$. If $M_* \sim 4\pi m_W$, our quantum field theoretic description of elementary particles is unlikely to be valid at energies above M_*, and thus there is no hierarchy problem provided M_* is not too much larger than the weak scale.

The known mechanisms for diluting the apparent strength of gravity typically make use of extra spacelike dimensions of spacetime.[1] Such extra dimensions appear to be an essential component of string theories [1, 2, 3], and they have been studied in various other contexts as well [4]. In the *Large Extra Dimensions* (LED) scenario, the strength of gravity we see is reduced by a factor of the volume of the extra dimensions [5, 6, 7]. With a *Warped Extra Dimenson*, gravity appears extremely weak to us because is localized away from us in the extra dimension [8, 9]. We will discuss both of these scenarios in these notes, as well as an intriguing connection between warped scenarios and strong coupling in four dimensions.

1 Large Extra Dimensions

Suppose we have n extra dimensions and let M_* be the fundamental Planck scale in the full $d = (4 + n)$-dimensional theory. Consider the gravitational potential Φ in the weak (Newtonian) limit. It satisfies

$$\vec{\nabla}^2\Phi \;\sim\; \frac{1}{M_*^{2+n}}\,\rho\;, \tag{1}$$

where ρ is the local energy density. For a pair of static point masses separated by a distance r, this leads to a gravitational force of

$$F(r) \;\sim\; \frac{1}{M_*^{2+n}}\frac{m_1m_2}{r^{2+n}}\;. \tag{2}$$

In contrast to $d = 4$, the gravitational flux lines can now spread out in more ways leading to a faster decrease of the force with distance. This is clearly inconsistent with the observed behaviour of gravity.

Let us now modify this picture by taking the n extra dimensions to all be periodic with radius R. Thus, for every extra-dimensional coordinate w we have $w^a \sim w^a + 2\pi R$, $a = 1, 2, \ldots n$. The gravitational force law takes the same form as before provided $r \ll R$. However, for distances large compared to the radius of the extra dimensions we have

$$F(r) \;\sim\; \frac{1}{M_*^{2+n}}\frac{m_1m_2}{r^2(2\pi R)^n} \qquad (r \gg R)\;. \tag{3}$$

[1] Timelike extra dimensions lead to challenges with causality and such.

In this case, the extent to which flux lines can spread is limited by the finite size of the extra dimensions. Matching this expressio to the previous one, we find that

$$M_{\rm Pl}^2 = (2\pi R)^n M_*^{2+n} = V_n M_*^{2+n} \ , \tag{4}$$

where V_n is the total volume of the compact extra dimensions.

Together, Eqs. (3,4) show how the strength of the gravitational force we observe can be diluted by the volume of extra dimensions. The idea of Refs. [5, 6, 7] was to use this dilution to recast the hierarchy problem by making R large enough that $M_* \sim$ TeV.[2] For n extra dimensions of equal size, the required radius R is

$$2\pi R \ \simeq \ 10^{32/n} 10^{-17}{\rm cm} \ \sim \ \left\{ \begin{array}{ccccc} 10^{15}\,{\rm cm} & (R^{-1} \sim ...) & ; & n=1 \\ 1\,{\rm mm} & (R^{-1} \sim 10^{-13}\ {\rm GeV}) & ; & n=2 \\ 1\,\mu{\rm m} & (R^{-1} \sim 10^{-8}\ {\rm GeV}) & ; & n=3 \\ 10\,{\rm fm} & (R^{-1} \sim 10^{-2}\ {\rm GeV}) & ; & n=6 \end{array} \right. . \tag{5}$$

These radii are very large compared to typical particle physics scales. For this reason, scenarios of this type are referred to as *large extra dimensions* (LED) or ADD after the original authors [5, 6, 7]. In this section we will discuss the implications of LED models. We will go over the new particles they predict, and we will discuss the extent to which they are constrained by existing data and how they may be probed in the future.

1.1 Kaluza-Klein Modes

To discuss the implications of LED, we will need to study quantum fields defined in more than four dimensions. When the extra dimensions are compact, a single d-dimensional field can be reduced to a set of four-dimensional fields called *Kaluza-Klein* (KK) modes. We will show how such KK modes arise in this subsection within a simple scalar model.

For our notation, we will write $x^M = (x^\mu, w^a)$ for the $d = 4+n$ spacetime coordinates with $M = 0, 1, 2, \ldots, 3+n$ and $a = 1, \ldots n$. The full d-dimensional metric will be denoted by G_{MN} with the flat space limit being $\eta_{MN} = diag(+1, -1, \ldots, -1)$. Note as well that when there is only one extra dimension, it is common practice to use the indices $M = 0, 1, 2, 3, 5$.

Consider now a free real scalar field $\Phi(x, w)$ in $d = 5$ dimensions with the extra dimension periodic with radius R, $w \sim w + 2\pi R$. The basic action in a flat background is

$$S = \int d^4x \int_0^{2\pi R} dw \ \left[\frac{1}{2}\eta^{MN}\partial_M\Phi\partial_N\Phi - \frac{1}{2}m^2\Phi^2\right] \ . \tag{6}$$

Given the geometry of the extra dimension, the field should also be periodic: $\Phi(x, w+2\pi R) = \Phi(x, w)$. This implies that the field can be expanded in a set of orthonormal basis functions $\{f_n\}$ according to

$$\Phi(x, w) = \sum_n f_n(w)\, \phi^{(n)}(x) \ , \tag{7}$$

[2] This isn't quite a solution to the hierarchy problem unless a mechanism to fix the radii of the extra dimensions is specified [10].

where the coefficients $\phi^{(n)}(x)$ depend only on the usual spacetime dimensions and

$$\int_0^{2\pi R} dw\, f_m(w)\, f_n(w) = \delta_{mn} \ . \tag{8}$$

The appropriate basis functions in this case are

$$f_n(w) = \frac{1}{\sqrt{2\pi R}} e^{inw/R} \ , \qquad n \in \mathbb{Z} \ . \tag{9}$$

These functions are clearly periodic and orthonormal. Since $\Phi(x, w)$ is real-valued, we must also have $\phi^{(n)\dagger}(x) = \phi^{(-n)}(x)$.

Plugging the expansion of Eq. (7) back into the action and using orthonormality, we get

$$\begin{aligned} S &= \int d^4x \, \Bigg(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^{(0)}\partial_\nu\phi^{(0)} - \frac{1}{2}m^2\phi^{(0)2} \\ &\quad + \sum_{n=1}^{\infty} \left[\eta^{\mu\nu}\partial_\mu\phi^{(n)\dagger}\partial_\nu\phi^{(n)} - (m^2 + \frac{n^2}{R^2})|\phi^{(n)}|^2 \right] \Bigg) \ . \end{aligned} \tag{10}$$

This is a four-dimensional theory containing a single real scalar $\phi^{(0)}$ of mass m and a tower of complex scalars $\phi^{(n)}$ with masses $m_n = \sqrt{m^2 + n^2/R^2}$, $n = 1, 2, \ldots$. The lightest state is called a *zero mode*, while the $n \geq 1$ fields are the KK modes.

The KK modes would be massive even in the limit of $m^2 \to 0$, and the extra contributions to their masses can be related to a quantized momentum in the fifth dimension. This can be seen by noting that the expansion of Eq. (7) is just a Fourier transform of the extra dimension for a periodic space. A further confirmation of this property can be found by examining a non-trivial interaction:

$$\begin{aligned} S &\to S_{free} - \int d^4x \int dw\, \lambda_5 \Phi^4 \\ &= S_{free} - \int d^4x \, \frac{\lambda_5}{2\pi R} \sum_{k,l,m,n} \phi^{(k)}\phi^{(l)}\phi^{(m)}\phi^{(n)}\, \delta_{k+l+m+n,0} \ . \end{aligned} \tag{11}$$

This interaction conserves momentum in the extra dimension at the vertex. Note as well that the theory is non-renormalizable since λ_5 has mass dimension of minus one (so $\lambda_5/2\pi R$ is dimensionless), but the theory is still predictive for $E \ll \lambda_5^{-1} \lesssim 4\pi/R$, where the second inequality arises from the requirement that the effective coupling of the KK modes is perturbative.

It is straightforward to generalize the KK expansion to more compact extra dimensions. For example, with n periodic extra dimensions of radius R, the basis functions would be

$$f_{\vec{n}}(\vec{w}) = \frac{1}{(2\pi R)^{n/2}}\, e^{i\vec{n}\cdot\vec{w}/R} \ , \qquad \vec{n} \in \mathbb{Z}_n \ , \tag{12}$$

where $\vec{w} = (w^1, \ldots, w^n)$ and $\vec{n} = (n^1, \ldots, n^n)$. The corresponding KK masses in this case are $m_{\vec{n}}^2 = m^2 + \vec{n}^2/R^2$. (Sorry about all the n's here.)

1.2 Gravitons in LED

Let us turn next to realistic particle theories in LED. We will assume there are n periodic extra dimensions of radius R. If we were to put the SM in the full extra dimensional space, every SM particle would have KK modes separated in mass by $1/R$. Since $1/R$ is very small by particle physics standards in LED theories (for n not too large), such KK modes would already have been observed if they existed. Instead, the SM fields are assumed to be restricted to a four-dimensional subsurface of the full spacetime. This is found to occur in many cases in string theory, where fields can be confined to dynamical subsurfaces called *branes* [11]. The only field that propagates within the full spacetime in LED models is the graviton, which therefore develops KK modes.

The graviton in LED emerges from expanding the metric around a background spacetime, which we will assume to be flat,

$$G_{MN} \;=\; \eta_{MN} + h_{MN}/M_*^{1+n/2} \;, \tag{13}$$

where the factors of M_* ensure that h_{MN} has the correct dimensions for a bosonic field in $d = 4 + n$ dimensions. The corresponding graviton action is

$$S_{grav} = \frac{M_*^{n+2}}{2}\int d^4x \int d^n w\, \sqrt{|G|}\, R^{(d)} \;, \tag{14}$$

where $R^{(d)}$ is the d-dimensional Ricci tensor built from G_{MN}. Expanding out the metric in this expression produces kinetic terms for h_{MN} as well as self-interactions. We can rewrite the action in terms of a set of KK modes by expanding in terms of basis functions,

$$h_{MN}(x,w) = \sum_{\vec{n}} \frac{1}{(2\pi R)^{n/2}}\, h^{(\vec{n})}_{MN}(x)\, e^{i\,\vec{n}\cdot\vec{w}/R} \;. \tag{15}$$

Putting this back into Eq. (14), the integration over the extra dimensions can be performed explicitly.

Compared to the simple scalar theory presented above, there is a new twist to the graviton [12, 13, 14]. It has multiple components related to the spacetime indices it carries, and these components transform in different ways under the four-dimensional Lorentz subgroup of the full spacetime coordinate invariance. Starting with $h_{MN}(x,w)$, it is a two-index symmetric tensor with real entries. This would give $d(d+1)/2$ degrees of freedom, but some of these turn out to be related by the underlying invariance under general coordinate transformations.

$$G_{MN} \to G'_{PQ} = \frac{\partial x^M}{\partial y^P}\frac{\partial x^N}{\partial y^Q}\, G_{MN} \;. \tag{16}$$

For an infinitesimal coordinate transformation $y^M = x^m + \xi^M$, this implies that

$$h_{MN} \to h'_{PQ} = h_{PQ} - \partial_P \xi_Q - \partial_Q \xi_P \;. \tag{17}$$

We can use this invariance to project out the redundant components. A popular choice is harmonic gauge

$$\partial_M h^M_{\ N} = \frac{1}{2}\partial h^M_{\ M} \ , \tag{18}$$

which imposes d conditions. This gauge still allows residual transformations with $\partial^2 \xi_M = 0$, and this fixes an additional d components. Together, we have a total of $d(d+1)/2 - 2d = d(d-3)/2 = (n+1)(n+4)/2$ independent components.

The next step is to decompose the KK modes of the graviton into quantities with well-defined transformations of the four-dimensional Lorentz subgroup. At each KK level, we have [12, 13, 14]

$$h^{(\vec{n})}_{MN} \ \rightarrow \ \left\{ \begin{array}{l} h^{(\vec{n})}_{\mu\nu} \\ h^{(\vec{n})}_{\mu a} = V^{(\vec{n})}_{\mu a} \\ h^{(\vec{n})}_{ab} = S^{(\vec{n})}_{ab} \end{array} \right. \ . \tag{19}$$

The zero modes $\vec{n} = \vec{0}$ are massless and independent of $\vec{w}$. Counting them, we have two degrees of freedom for the massless graviton, $2n$ degrees of freedom for the massless vectors, and $n(n+1)/2$ for the massless scalars to give $(n^2+5n+4)/2$ in total, as expected. The KK modes have masses $m^2_{\vec{n}} = \sqrt{\vec{n}^2/R^2}$. At level $\vec{n}$, there is one massive graviton with five degrees of freedom, $(n-1)$ massive vectors with three degrees of freedom each, and $n(n+1)/2 - n$ massive scalars for a total of $(n^2 + 4n + 5)$. In counting these, we have made use of the constraints implied by general coordinate invariance which translate into the conditions

$$n^a V^{(\vec{n})}_{\mu a} = 0 \ , \qquad n^a S^{(\vec{n})}_{ab} = 0 \ . \tag{20}$$

These can also be understood in that at each KK level the massless graviton eats a massless vector and a massless scalar to get a mass, while the remaining $(n-1)$ massless vectors each eat a scalar.

With these modes in hand, we would like to figure out how they couple to the SM fields. Recall that the SM is assumed to be confined to a four-dimensional brane, and we can choose its location to be $w^a = 0$. The coupling to gravity is the usual minimal form, but now with an explicit localization,

$$S_{SM} \ = \ \int d^4x \int d^n w \ \sqrt{|G|}\, \mathscr{L}_{SM}\, \delta^{(n)}(\vec{w} - \vec{0}) \tag{21}$$

$$= \ \int d^4x \ \sqrt{|g|}\, \mathscr{L}_{SM} \tag{22}$$

where $g_{\mu\nu}(x) = G_{\mu\nu}(x, w = 0)$ is the *induced metric*. Expanding this out, one obtains couplings of the form [12, 13]

$$S_{SM} \ \supset \ -\int d^4x \, \frac{1}{M_*^{1+n/2}\sqrt{V_n}} \left[T^{\mu\nu} \sum_{\vec{n}} h^{(\vec{n})}_{\mu\nu} - \kappa T^{\mu}_{\ \mu} \sum_{\vec{n}} S^{(\vec{n})\ a}_{a} \right] \ , \tag{23}$$

where κ_n is an n-dependent constant of order one and

$$T^{\mu\nu} = \frac{1}{\sqrt{|g|}}\frac{\delta S_{SM}}{\delta g_{\mu\nu}} \tag{24}$$

is the energy-momentum tensor of the SM. The first term reproduces the usual coupling of the massless graviton (zero mode) to the SM once we identify $M_*^{n+2}V_n = M_{\rm Pl}^2$, as well as couplings of the SM to the KK gravitons. The second term connects the SM to a specific linear combination of the scalars called the *radion* and its KK modes. Note that none of the other components of the graviton zero or KK modes couple to the SM. For this reason, they can (mostly) be ignored, and the only graviton excitations that need to be considered are: the massless $h_{\mu\nu}^{(0)}$ and $r^{(0)} = S_a^{(\vec{0})\ a}$ zero modes; and the massive $h_{\mu\nu}^{(\vec{n})}$ and $r^{(0)} = S_a^{(\vec{n})\ a}$ KK modes.

The existence of a massless radion zero mode is a problem since it would modify gravity at long distances. In this context, the masslessness of the radion correponds to a flat "potential" for expanding or shrinking the radius R of the extra dimensions. A stabilization mechanism is needed to fix this, and the massless radion is expected to develop a mass as a result [10].

1.3 Experimental Tests and Constraints

The new graviton KK modes predicted by LED models are constrained and are being looked for in a number of ways. The strongest bounds typically come from deviations from $1/r^2$ gravity and modifications to stellar evolution, while searches for the new KK states are underway at the LHC. Limits on LED are usually quoted in terms of a lower bound on M_* for a given number of extra dimensions n. Recall that we want $M_* \lesssim$ TeV for this scenario to address the electroweak hierarchy problem. A more detailed discussion can be found in Ref. [14].

Light graviton (or radion) KK modes can modify the $1/r^2$ behaviour of the gravitational force at short distances, $r \lesssim R$. Such deviations have been investigated, and it is standard practice to parametrize them according to

$$V(r) = -\frac{1}{8\pi M_{\rm Pl}^2}\frac{m_1 m_2}{r}\left(1+\alpha e^{-r/\lambda}\right) \ , \tag{25}$$

where λ and α are dimensionless parameters. For the first deviations from massive KK gravitons, we expect $\alpha = 1$ and $\lambda = R$. The current limit for $\alpha = 1$ is $\lambda \leq 44\,\mu$m [15]. This rules out $n = 1$, and forces $M_* \gtrsim 1.4$ TeV for $n = 2$, but does not provide a useful constraint for $n \geq 3$.

The LED theory with KK modes at $M_* \sim$ TeV only works as an effective theory valid at energies below $E \lesssim M_*$, and the physics above this is unknown. This new physics, which includes quantum gravity, is likely to generate non-renormalizable operators involving SM fields such as

$$-\mathscr{L} \supset \frac{1}{M_*^2}(\bar{f}_1\Gamma f_2)(\bar{f}_3\Gamma' f_4) \ , \tag{26}$$

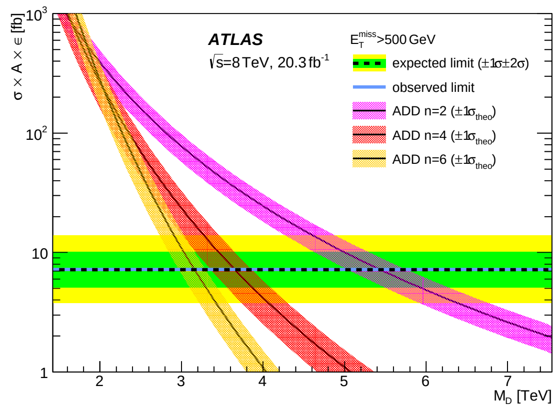

Figure 1: . [17]

where $f_{1,2,3,4}$ are SM fermions, and Γ and Γ' are Dirac structures. If such operators violate baryon and lepton number, limits on nucleon decay force $M_* \gtrsim 10^{16}$ GeV. If they generate new flavour mixing of CP violation, $M_* \gtrsim 10^6$ GeV. And even if they respect all the symmetries of the SM, precision electroweak constraints limit $M_* \gtrsim 10$ TeV. The suppression of these operators cannot be addressed within the LED effective theory, but they are a cause for concern.

Collider experiments have also searched for the direct production of KK modes. These modes would typically escape the detector without leaving a trace, and they would therefore contribute to missing energy. The coupling of the SM to any single mode is tiny, suppressed by M_{Pl}, but there are many modes to sum over, and this makes the effective cross section potentially observable. Once all the accessible modes are summed over, the cross section scales like $1/M_*^2$ rather than $1/M_{\text{Pl}}^2$. Searches at the LHC often focus on events with a single hard jet and a large amount of missing energy, and current data limits $M_* \gtrsim 5.5,\ 4.3,\ 3.2$ TeV for $n = 2, 3, 6$, as can be seen in Fig. 1 [16, 17].

The many light KK modes predicted by LED theories can also contribute to astrophysical processes. In particular, they can cause supernovae to cool more quickly that they would otherwise. This can occur through the production of KK gravitons in the high-energy particle collisions, corresponding to temperatures of about $T \simeq 50$ MeV, which then escape the supernova envelope. The strongest limits come from observations of SN1987A, and give $M_* \gtrsim 50,\ 4,\ 1$ TeV for $n = 2, 3, 4$ [18]. For $n > 4$, there is no limit because the SN temperature is not high enough to produce the heavier KK gravitons efficiently.

2 Warped Extra Dimensions

A second approach to diluting the effective strength of gravity makes use of localization within an extra dimension.

3 Warping and Strong Coupling

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