PSI BSM Homework #3

1. **Bulk Scalars in a Warped XD**
   Consider a the warped RS geometry of

$$ds^2 = \left(\frac{L_0}{z}\right)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right) = G_{MN} dx^M dx^N \ ,$$

   with $z \in [L_0, L_1]$ and $L_0 \ll L_1$. The minimal action for a scalar propagating in the bulk of this spacetime is

$$S = \int d^4x \int_{L_0}^{L_1} dz \sqrt{|G|}\, \frac{1}{2} G^{MN} \partial_M \phi \partial_N \phi \ .$$

   a) Expand out the Lagrangian, integrate by parts (assuming you can drop all boundary terms), and show that the classical equation of motion is

$$\left[-\partial^2 + \partial_z^2 - (3/z)\partial_z\right] \phi = 0 \ .$$

   Make sure to show how you got your result.

   b) Show that if we expand $\phi(x, z)$ in a set of basis functions according to,

$$\phi(x, z) = \sum_n \phi^{(n)}(x) f_n(z) \ ,$$

   such that

$$\begin{aligned} \left[\partial_z^2 - (3/z)\partial_z\right] f_n &= -m_n^2 f_n \ , \\ \int_{L_0}^{L_1} dz \left(\frac{L_0}{z}\right)^3 f_m f_n &= \delta_{mn} \ , \end{aligned}$$

   we obtain the action for a $d = 4$ theory with a tower of scalars of mass m_n.

   c) Solve for these basis functions when $m_n^2 \neq 0$. For this, write $f(z) = z^\alpha g(z)$ and show with a clever choice of α, the differential equation for $g(z)$ becomes a Bessel equation,

$$\left[z^2 \partial_z^2 + z\partial_z - (m_n^2 z^2 - \nu^2)\right] g = 0 \ ,$$

   for some constant ν^2. Recall that the most general solutions of this equation are the Bessel functions $J_\nu(m_n z)$ and $Y_\nu(m_n z)$.

   d) The most general solution for $m_n \neq 0$ is therefore

$$f_n(z) = N_n \left(\frac{z}{L_0}\right)^\alpha \left[J_\nu(m_n z) + \beta_n Y_\nu(m_n z)\right] \ .$$

   Solve for β_n and find the possible mass eigenvalues m_n assuming Neumann boundary conditions.
   Hint: use the nice properties of Bessel functions to make this easier. In particular, $(1/z)\partial_z(z^a J_a) = z^{a-1} J_{a-1}$ and similarly for Y_a.

e) The zero-mode case is special. For this, solve for $g(z)$ when $m_n = 0$. Show that there is a zero mode for Neumann BCs but not Dirichlet.

f) Suppose we now add a bulk mass term $m^2\phi^2/2$ to the Lagrangian. Solve for the zero mode profile for both Neumann and Dirichlet BCs.

2. **Yukawa Couplings in RS with Bulk Fermions**
Fermions are chiral in $d = 5$, but we can arrange for them to have chiral zero modes by an appropriate choice of boundary conditions. Given the bulk fermion $\Psi = (\psi_L, \psi_R)^t$, choosing the BC $\psi_R|_{z=L_0,L_1} = 0$ yields only a left-handed zero mode with profile

$$f_L^{(0)}(z) = N_0 \left(\frac{z}{L_0}\right)^{2-c} .$$

Similarly, if we choose the BC $\psi_L|_{z=L_0,L_1} = 0$, there is only a righ-handed zero mode with profile

$$f_R^{(0)}(z) = N_0' \left(\frac{z}{L_0}\right)^{2+c} .$$

a) Compute the normalizations N_0 and N_0' following from the requirement that

$$1 = \int_{L_0}^{L_1} dz \left(\frac{L_0}{z}\right)^4 \left[f_{L,R}^{(n)}\right]^2 .$$

b) The term in the action connecting the bulk fermions Q and U to the Higgs is

$$S \supset \; -\int d^4x \int_{L_0}^{L_1} \sqrt{|G|}\, \left[\left(\frac{z}{R}\right) y\, H\, \overline{Q}_L U_R + (h.c.)\right] \delta(z - L_1) .$$

How should the BCs of Q and U be chosen to reproduce the SM Yukawa interaction in terms of the zero modes? What is the effective Yukawa coupling in terms of y and the c_Q and c_U coefficients determining the profiles of Q and U?

3. **QCD Axions**
This is a topic we didn't have time to cover, but you have enough background material to learn about them in this problem. Suppose $\Psi = (\Psi_L, \Psi_R)$ is an exotic quark that is a triplet under $SU(3)_c$. Through a subtle QFT effect, it can be shown that the transformation

$$\begin{aligned} \Psi_L(x) &\rightarrow e^{i\phi_L}\Psi_L \\ \Psi_R(x) &\rightarrow e^{i\phi_R}\Psi_L \end{aligned}$$

implies that $\Theta \rightarrow \Theta + (\phi_L - \phi_R)$, where Θ is the so called *theta term* of QCD:

$$\mathscr{L}_{QCD} \supset \left(\frac{\alpha_s}{8\pi}\right) \Theta\, \widetilde{G}^a_{\mu\nu} G^{a\mu\nu} \ .$$

The *strong CP problem* is that $|\Theta| \lesssim 10^{-10}$ from experiment, but there is no obvious reason why it should be so small. A nice way to address the problem is by introducing an axion field. In this problem, you will work out a simple axion model.

a) Consider a coupling between Ψ and the complex scalar ϕ

$$\mathscr{L} = |\partial\phi|^2 - V(\phi) - y\left(\phi\bar{\Psi}_L\Psi_R + \phi^*\bar{\Psi}_R\Psi_L\right) \ ,$$

with $V(\phi) = \lambda(|\phi|^2 - f_a^2)^2$. Show that $\langle\phi\rangle = f_a$ minimizes the potential.

b) Expand about the vacuum by writing $\phi(x) = \frac{1}{\sqrt{2}}(f_a + h(x))e^{ia(x)/f_a}$. Show that this gives canonical kinetic terms for both a and h by plugging the expansion back into the Lagrangian.

c) Derive the masses induced for h, a, and Ψ. You should find that a is massless. *Hint: expand* $e^{ia/f} = 1 + ia/f_a + \ldots$.

d) For $f_a \gg m_W$, we can integrate out the heavy particles from the theory to get an effective theory with only the light states. Show that by redefining the fields Ψ_L and Ψ_R by

$$\begin{aligned} \Psi_L(x) &\rightarrow \Psi_L\, e^{iQ_L a(x)/f_a} \ , \\ \Psi_R(x) &\rightarrow \Psi_R\, e^{iQ_R a(x)/f_a} \ , \end{aligned}$$

we can remove the direct coupling of a to the fermions if we take $(Q_L - Q_R) = 1$. Show as well that this induces a coupling of a to $G^a_{\mu\nu}\widetilde{G}^{a\,\mu\nu}$ and find the coupling coefficient of the effective operator.

e) At lower energies approaching Λ_{QCD}, the gluon operator $G\widetilde{G}$ develops a condensate that generates a potential for $a(x)$,

$$V(a) \simeq m_a^2 f_a^2 \left[1 - \cos(\Theta + a/f_a)\right] \ .$$

Minimize the potential and show that the total effective theta term is cancelled off at the minimum. Also, expand around the minimum to find the mass of the remaining physical excitation.