PHYS 528 Homework #10

Due: Mar.30, 2017

1. QFT practice #1.

   Recall that the standard expansion for free a Dirac fermion is

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{s=1,2} \left[u(k,s) b_{\vec{k},s}\, e^{-ik\cdot x} + v(k,s) d^\dagger_{\vec{k},s} e^{ik\cdot x} \right] \ ,$$

   where $k^0 = E_k = \sqrt{m^2 + \vec{k}^2}$, $u(k,s)$ and $v(k,s)$ are the usual Dirac spin vectors for spin state s, and the raising and lowering operators satisfy

$$\{b_{\vec{k},s},\, b^\dagger_{\vec{p},r}\} \ = \ (2\pi)^3 2E_k\, \delta^{(3)}(\vec{k} - \vec{p})\, \delta_{rs} \ = \ \{d_{\vec{k},s},\, d^\dagger_{\vec{p},r}\} \ ,$$

   with all other anticommutators vanishing. We identify $b^\dagger_{\vec{k},s}$ as the creation operator for a fermion of type ψ with 3-momentum $\vec{k}$ and spin state s, and $d^\dagger_{\vec{k},s}$ as the creation operator for the corresponding anti-fermion $\bar{\psi}$. External momentum states correspond to these operators acting on the vacuum:

$$|\psi(\vec{k}, s)\rangle = b^\dagger_{\vec{k},s}|0\rangle \ , \qquad |\bar{\psi}(\vec{k}, s)\rangle = d^\dagger_{\vec{k},s}|0\rangle \ .$$

   a) Compute $\langle 0|\psi(x)|\psi(\vec{p}, r)\rangle$, $\langle 0|\bar{\psi}(x)|\bar{\psi}(\vec{p}, r)\rangle$, $\langle \psi(\vec{p}, r)|\bar{\psi}(x)|0\rangle$, and $\langle \bar{\psi}(\vec{p}, r)|\psi(x)|0\rangle$, and match up your results with the standard Feynman rules for external fermions.

   b) If the theory has a $U(1)$ symmetry under rephasing ψ, the conserved current is proportional to $j^\mu(x) = \bar{\psi}\gamma^\mu\psi$. Define $\tilde{j}^\mu(p) = \int d^4x\, e^{-ip\cdot x} j^\mu(x)$ and compute $\langle 0|\tilde{j}^\mu(q)|\psi(k, s), \bar{\psi}(p, r)\rangle$. What would a Feynman diagram for this look like? Show also that all matrix elements of $\tilde{j}^\mu(p)$ between the vacuum and one-particle fermion initial states vanish.

2. QFT practice #2.

   The discussion above carries over to a complex scalar $\phi(x)$ if we remove the spin stuff.

   a) Calculate $\langle 0|\phi(x)|\phi(\vec{p})\rangle$, $\langle 0|\phi^\dagger(x)|\phi^*(\vec{p})\rangle$, $\langle \phi(\vec{p})|\phi^\dagger(x)|0\rangle$, and $\langle \phi^*(\vec{p})|\phi(x)|0\rangle$.

   b) If ϕ has a symmetry under rephasing, the corresponding current is proportional to $j^\mu(x) = -i\phi^* \overleftrightarrow{\partial}_\mu \phi$. Evaluate $\langle 0|\tilde{j}^\mu(p)|\phi(\vec{k})\phi^*(\vec{q})\rangle$.

   c) Suppose the rephasing symmetry of ϕ is spontaneously broken, with $\langle \phi(x)\rangle = v$ at the minimum of the potential. It is now convenient to rewrite the field as

$$\phi(x) = (v + h/\sqrt{2})e^{i\rho(x)/\sqrt{2}v} \ ,$$

   where $h(x)$ and $\rho(x)$ are real scalar fields. In this case, the physical excitations can be identified with $h(x)$ and $\rho(x)$ with self-conjugate mode expansions of the form $\rho(x) = \int [d^3k/2E_k(2\pi)^3] \left(a_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{ik\cdot x} \right)$ and states $|\rho(\vec{p})\rangle = a^\dagger_{\vec{p}}|0\rangle$. Compute the matrix element $\langle 0|\tilde{j}^\mu(p)|\rho(\vec{k})\rangle$.

3. **Charged Pions**

Let us examine some of the claims we made about pions in two-flavour QCD:

a) We stated that the QED generator was

$$Q = t^3_L + t^3_R + \mathbb{I}/6 \ .$$

Acting on the field Σ, this means that we set $c^a_L = \delta^{a3}\alpha = c^a_R$ and $\alpha_V = \alpha/6$ (for $L = \exp(ic^a_L t^a)$, $R = \exp(ic^a_R t^a)$, and $V = \exp(i\alpha_V)$) for some transformation parameter α, so that finite QED transformations take the form

$$\Sigma \to e^{i\alpha t^3} e^{i\alpha/6}\, \Sigma\, e^{-i\alpha t^3} e^{-i\alpha/6} \ .$$

Work out what this implies for the transformation properties of the Σ and Π^a fields to linear order in α.

b) Define $\pi^0 = \Pi^3$ and $\pi^\pm = (\Pi^1 \mp i\Pi^2)/\sqrt{N}$.

i) Express $\Pi^a t^a$ in terms of π^0, $\pi^\pm$ and combinations of the t^a.

ii) Show that $[t^3, t^1 \mp it^2] = (\mp)(t^1 \mp it^2)$.

iii) Use these results to figure out the infinitesimal QED transformations of the π^0 and $\pi^\pm$ fields.

c) To incorporate electromagnetism (EM) into our pion theory, we need to add a photon field and upgrade the regular derivatives on Σ to covariant derivatives. Recall that for a field ψ with $U(1)_{em}$ charge q, the transformations

$$\psi \to e^{iq\alpha(x)}\psi = (1 + iq\alpha + \ldots)\psi \ , \qquad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha \ ,$$

imply that $D_\mu\psi = (\partial_\mu + iqeA_\mu)\psi \to e^{iq\alpha}D_\mu\psi$. By analogy to this form, and using the infinitesimal EM transformation on Σ you found above, construct a covariant derivative for Σ and use this to build the leading term in the chiral Lagrangian that is $SU(2)_L \times SU(2)_R$-invariant for $e \to 0$. Check that it is indeed covariant to leading non-trivial order in $1/f$.

Hint: note the connection between the infinitesimal transformation of the field and the covariant derivative and make use of it. Also, commutators.

d) Show that gauging only a subgroup of the global flavour group G_{flav} explicitly breaks the invariance under G_{flav}. To do so, apply an infinitesimal t^1_L or t^2_L rotation to the theory and show that the photon term in the covariant derivative messes up the invariance. What happens if you apply an infinitesimal t^3_L transformation instead?

4. Read `notes-10`.
Did you read them? (Y/N)