# PHYS 528 Homework #1

Due: Jan. 19, 2017

1. Consider a theory with two real scalar fields and the Lagrangian

$$\mathscr{L} = \frac{1}{2} Z_{ij} \eta^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_j - \frac{1}{2} M^2_{ij} \phi_i \phi_j \ ,$$

   where $i, j = 1, 2$, and

$$Z = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} , \qquad M^2 = m^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

   Find the masses of the physical excitations in the theory.

2. Fun with the action.

   a) For the scalar action

$$S[\phi] = \int d^4x \, \left[ \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 \right] \ ,$$

   integrate the kinetic term by parts so that both derivatives are acting on a single field operator. Also, evaluate the action for the specific field configuration $\phi(x) = a(\vec{k}) \exp(-ik \cdot x)$ for an arbitrary 4-vector $k = (k^0, \vec{k})$ and function $a(\vec{k})$. Show that it vanishes for $k^0 = \pm\sqrt{m^2 + \vec{k}^2}$.

   b) Do all the same things for the vector boson action

$$S[A^\mu] = \int d^4x \, \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right] \ .$$

   c) Show that the basic action for a Dirac fermion is real: $S = S^*$ for

$$S[\psi, \overline{\psi}] = \int d^4x \, \overline{\psi} (i\gamma^\mu \partial_\mu - m) \psi \ .$$

   *Hint:* $(\overline{\psi}_1 \psi_2)^* = (\overline{\psi}_1 \psi_2)^\dagger = (\psi_1^\dagger \gamma^0 \psi_2)^\dagger = \psi_2^\dagger (\gamma^0)^\dagger \psi_1$.

3. Work out the differential cross section $d\sigma/d(\cos\theta)$ for the process $e^+ e^- \to \mu^+ \mu^-$, where $\theta$ is the CM-frame angle between the incident electron and the outgoing muon. You may work in the limit that $p^2 \gg m_e^2$ and ignore the electron mass ($m_e \to 0$), but do keep the full dependence on the muon mass.
   *Hint: for the integrals over phase space, use the spatial components of the overall delta function to get rid of the $d^3 p_4$ integral, and then use the remaining time (energy) component to get rid of the integration over the magnitude of $\vec{p}_3$. For this, you'll probably want to use the relation $\int dx \, \delta(f(x)) g(x) = g(x_0)/|df/dx|_{x=x_0}$, where $x_0$ is the value of $x$ such that $f(x_0) = 0$.*

4. Compute the differential and total cross sections for $e^- \mu^- \to e^- \mu^-$ scattering to leading order in QED at very high energy, $E_{CM} \gg m_\mu$, $m_e$. This implies that you can neglect the fermion masses.

5. Consider a massive $Z'$ vector boson that couples to muons with a vertex factor equal to $-ig'\gamma^\mu$.

   a) A massive vector has three independent polarization states. These can represented by any three independent unit 4-vectors $\epsilon_\mu(p, \lambda)$ satisfying the constraints $p^\mu \epsilon_\mu = 0$ and $\epsilon^*_\mu(p, \lambda)\epsilon^\mu(p, \lambda') = -\delta_{\lambda\lambda'}$, where $p^\mu$ is the four-momentum of the vector boson and $\lambda = 1, 2, 3$ labels the three different polarizations. Find a simple set of polarization vectors in the rest frame of the massive vector. Show that they satisfy the completeness relation

$$\sum_\lambda \epsilon_\mu(p, \lambda)\epsilon^*_\nu(p, \lambda) = -\eta_{\mu\nu} + p_\mu p_\nu / m^2_{Z'}.$$

   b) Compute the total unpolarized decay width for $Z' \to e^+e^-$. For this, use the vertex and the completeness relation stated above. Keep the full dependence on the masses of the $Z'$ and the electron.