PHYS 528 Lecture Notes #2

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1 Symmetries in QFT

Symmetries play a central role in modern particle physics. Insofar as we believe that elementary particles can be described by QFT (and the evidence so far points in this direction), our role as theoretical and experimental particle physicists is to figure out the Lagrangian of our world. In particular, we must specify a set of fields and their interactions. Once we have a candidate Lagrangian, we can compute the dynamics of the theory and compare to experiment. Symmetries make the task of figuring out the Lagrangian much easier because they strongly constrain the set of possible fields and interactions. They are also enormously useful in computing the dynamics.

To begin, let's look at a few simple examples:

***e.g.* 1.a)** $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

This theory is symmetric under $\phi \to -\phi$ in that the form of the Lagrangian doesn't change. The implication of this symmetry is that for any process, the number of particles in the initial state minus the number in the final state must be even.

***e.g.* 1.b)** $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}i\gamma^\mu\partial_\mu\psi - y\phi\bar{\psi}\psi$

This theory is symmetric under $\phi \to -\phi$, $\psi \to \gamma^5\psi$. (Note that the second condition implies $\bar{\psi} \to -\bar{\psi}\gamma^5$.) This symmetry forbids a fermion mass term. Such symmetries are sometimes called *chiral* symmetries.

***e.g.* 2.** $\mathscr{L} = |\partial\phi|^2 - M^2|\phi|^2 + \sum_{i=1}^{2}\bar{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - (y\phi\bar{\psi}_1\psi_2 + h.c.)$

This theory is symmetric under $\psi_1 \to e^{i\alpha Q_1}\psi_1$, $\psi_2 \to e^{i\alpha Q_2}\psi_2$, $\phi \to e^{i\alpha Q_\phi}\phi$ for any real constant α provided $(Q_\phi - Q_1 + Q_2) = 0$. These Q's are sometimes called the *charges* of the fields under the symmetry. In contrast to the previous examples, this symmetry is *continuous* rather than *discrete*. For generic Q's, the symmetry forbids a cross mass term of the form $\bar{\psi}_1\psi_2$, and interactions such as ϕ^2, ϕ^3, ϕ^4, $\phi\bar{\psi}_1\psi_1$. Note also that we can write the complex scalar field ϕ in terms of two real scalars, $\phi = (\phi_r + i\phi_i)/\sqrt{2}$. For $Q_\phi \neq 0$ both real scalars must have the same mass, and it turns out to be easier to treat them as a single complex scalar with distinct particle and antiparticle excitations, rather than two separate real scalars each of which is its own antiparticle.

To be more precise, we say that a transformation of the fields is a symmetry of the theory if the equations of motion for the transformed fields take the same form as those of the original fields. This is equivalent to having, under $\phi \to \phi'(\phi)$ and $S[\phi] \to S[\phi'(\phi)] \equiv S'[\phi]$, $S'[\phi] = S[\phi]$ up to total derivatives. In other words, the action retains the same functional form after the transformation (up to possible total derivatives).

At this point, it is worth remembering how one derives the equations of motion from an action. The solution to the equation of motion is the field configuration such that for any arbitrary infinitesimal field variation that vanishes on the boundary, $\phi \to \phi + \delta\phi$, the numerical value of the action remains the same to leading order: $\delta S = 0$. For a local QFT, this leads to

$$\delta S = \int d^4x \left(\left[\frac{\partial \mathscr{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} \right] \delta\phi_i + \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right] \right). \tag{1}$$

The term in the first bracket gives the equation of motion for ϕ_i by forcing it to vanish, while the second term is zero automatically because the field variation is assumed to vanish on the boundary.

In deriving the equation of motion above we considered arbitrary field variations (that vanish on the boundary). Symmetries, on the other hand, correspond to very specific field variations. In order to retain the same form of the equations of motion under a symmetry transformation, the Lagrangian can only change by a total derivative. In the case of a continuous symmetry, this implies the existence of a conserved charge. To see this, consider an infinitesimal symmetry transformation of the form

$$\phi_i \to \phi_i' = \phi_i + \delta\alpha^a F_{a,i}(\phi) \tag{2}$$

To leading order in $\delta\alpha^a$, the implication of this being a symmetry of the theory leads to

$$\begin{aligned} \delta\mathscr{L} = \delta\alpha^a \partial_\mu \mathcal{J}^\mu_a &= \frac{\partial \mathscr{L}}{\partial \phi_i} \delta\alpha^a F_{a,i} + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu \left(\delta\alpha^a F_{a,i} \right) \\ &= \left[\frac{\partial \mathscr{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} \right] \delta\alpha^a F_{a,i} + \delta\alpha^a \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} F_{a,i} \right] \end{aligned} \tag{3}$$

The first term vanishes for fields that satisfy the equation of motion. Keeping the second term, we conclude that

$$\begin{aligned} 0 &= \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} F_{a,i} - \mathcal{J}^\mu_a \right] \\ &\equiv \partial_\mu j^\mu_a \end{aligned} \tag{4}$$

Thus the quantity j^μ_a is a conserved current corresponding to the continuous symmetry.[1] In terms of components, $j^\mu_a = (\rho_a, \vec{j}_a)$; a charge density and a spatial current. We can also define a conserved charge according to

$$Q_a = \int d^3x j^0_a. \tag{5}$$

As long as the current vanishes on the spatial boundary, we find that the the charge is constant in time, $\partial_t Q_a = 0$.[2]

[1] This result is known as Noether's theorem.

[2] A reasonable requirement for quantum fields is that $\phi(x) \to 0$, $\partial_\mu \phi(x) \to 0$ as $|\vec{x}| \to \infty$ for any finite t.

e.g. **3.a)** $\mathscr{L} = \frac{1}{2}(\partial\phi)^2, \qquad \phi \to \phi + \alpha$

The Lagrangian is invariant under this transformation, so we have

$$j^\mu = \partial_\nu \phi \, \eta^{\mu\nu} \tag{6}$$

It's easy to check that this current is conserved via the equation of motion.

e.g. **3.b)** $\mathscr{L} = |\partial\phi|^2 - V(|\phi|^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi, \qquad \phi \to e^{-iQ_\phi\alpha}\phi,\, \psi \to e^{-iQ_\psi\alpha}\psi$

This leads to the symmetry current

$$j_\mu = Q_\psi \bar{\psi}\gamma^\mu\psi + Q_\phi \phi^*(i \overset{\leftrightarrow}{\partial}_\mu)\phi. \tag{7}$$

Note that here $\overset{\leftrightarrow}{\partial}_\mu = (\overset{\rightarrow}{\partial}_\mu - \overset{\leftarrow}{\partial}_\mu)$

A particularly important conserved current is the *energy-momentum* tensor. For a generic Lagrangian with no explicit dependence on x^μ, the theory is symmetric under spacetime translations: $x^\nu \to x^\nu + \alpha^\nu$, $\phi(x) \to \phi(x+\alpha) \simeq \phi(x) + \alpha^\nu \partial_\nu \phi + \mathcal{O}(\alpha^2)$. Under such a shift, the Lagrangian only changes by a total derivative, $\mathscr{L} \to \mathscr{L} + \alpha^\nu \partial_\nu (\delta^\mu{}_\nu \mathscr{L}) + \mathcal{O}(\alpha^2)$. The corresponding conserved current is therefore

$$j^\mu{}_\nu \equiv T^\mu{}_\nu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathscr{L}. \tag{8}$$

The quantity $T^\mu{}_\nu$ is called the energy-momentum tensor of the theory. Here, μ labels the component of the current, while ν labels the translation direction. If this is confusing, don't worry. We can always arrange for $T^{\mu\nu}$ to be symmetric in its indices. We interpret T^{00} as the energy density, T^{0i} as the i-th momentum density, and so on.

e.g. **4.** $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$

This gives

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \mathscr{L} \tag{9}$$

In particular, $T_{00} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V$, gives a reasonable energy density.

2 Symmetries and Groups

Symmetry transformations obey the mathematical properties of a *group*, and it's worth spending a bit of time discussing them.[3] A group G is a set of objects together with a multiplication rule such that:

[3] Much of this discussion is based on Refs. [1, 2], both of which provide a much more detailed account of the topics covered here.

1. if $f,\, g \in G$ then $h = f \cdot g \in G$ (closure)
2. $f \cdot (g \cdot h) = (f \cdot g) \cdot h$ (associativity)
3. there exists an identity element $1 \in G$ such that $1 \cdot f = f \cdot 1 = f$ for any $f \in G$ (identity)
4. for every $f \in G$ there exists an inverse element f^{-1} such that $f \cdot f^{-1} = f^{-1} \cdot f = 1$ (invertability)

A group can be defined via a multiplication table which specifies the value of $f \cdot g$ for every pair of elements $f,\, g \in G$. An *Abelian* group is one for which $f \cdot g = g \cdot f$ for every pair of $f,\, g \in G$. A familiar example of an Abelian group is the set of rotations in two dimensions. In contrast, the set of rotations in three dimensions is non-Abelian.

For the most part, we will be interested in symmetry transformations that act linearly on quantum fields,

$$\phi_i \to \phi_i' = U_{ij}\phi_j. \tag{10}$$

As a result, we will usually work with matrix *representations* of groups. Groups themselves are abstract mathematical objects. A representation of a group is a set of $n \times n$ matrices $U(g)$, one for each group element, such that:

1. $U(f)U(g) = U(f \cdot g)$
2. $U(1) = \mathbb{I}$, the identity matrix.

Note that these conditions imply that $U(f^{-1}) = U^{-1}(f)$. The value of n is called the dimension of the representation. For any group, there is always the *trivial* representation where $U(g) = \mathbb{I}$ for every $f \in G$. Note that a representation does not have to faithfully reproduce the full multiplication table. A representation is said to be *unitary* if all the representation matrices can be taken to be unitary ($U^\dagger = U^{-1}$).

***e.g.* 1.** Rotations in two dimensions
This group is formally called $SO(2)$ and can be defined as an abstract mathematical object. Any group element can be associated with a rotation angle θ. The most familiar representation is in terms of 2×2 matrices,

$$D(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{11}$$

Of course, there is also the trivial representation.

Our focus will be primarily on continuous transformations. These correspond to what are called *Lie groups*, which are simply groups whose elements can be parametrized in terms of a set of continuous variables $\{\alpha^a\}$. We can (and will) always choose these coordinates (near the identity) such that the point $\alpha^a = 0$ corresponds to the identity element of the group. Thus, for any representation of the group, we have for infinitesimal transformations near the identity

$$U(\delta\alpha^a) = \mathbb{I} + i\delta\alpha^a t^a + \mathcal{O}(\delta\alpha^2). \tag{12}$$

The matrices t^a are called *generators* of the representation. Finite transformations can be built up from infinitesimal ones according to

$$U(\alpha^a) = \lim_{p\to\infty}(1 + i\alpha^a t^a/p)^p = e^{i\alpha^a t^a}. \tag{13}$$

This is nice because it implies that we only need to sort out a finite set of generators when discussing the representation of a Lie group rather than the infinite number of group elements.

A set of generator matrices $\{t^a\}$ can represent a Lie group provided they satisfy a *Lie algebra*. Besides being able to add and multiply them, they must also satisfy the following conditions:

1. $[t^a, t^b] = if^{abc}t^c$ for some constants f^{abc}
2. $[t^a, [t^b, t^c]] + [t^b, [t^c, t^a]] + [t^c, [t^a, t^b]] = 0$ (Jacobi Identity)

The first condition is needed for the closure of the group (*i.e.* $\exp(i\alpha^a t^a)\exp(i\beta^a t^a) = \exp(i\lambda^a t^a)$ for some λ^a) while the second is required for associativity. In fact, we can define a Lie group abstractly by specifying the *structure constants* f^{abc}. Most of the representations we'll work with are unitary, in which case the structure constants are all real and the generators t^a are Hermitian.

The nice thing about working with linear generators t^a is that we can choose a nice basis for them. This is equivalent to choosing a nice set of coordinates for the Lie group. In particular, it is always possible to choose the generators t^a_r of any representation r such that

$$tr(t^a_r t^b_r) = T_2(r)\delta^{ab}. \tag{14}$$

The constant $T_2(r)$ is called the Dynkin index of the representation. We will always implictly work in bases satisfying Eq. (14), and we will concentrate on the case where the index is strictly positive. If so, the corresponding Lie group is said to be *compact* and is guaranteed to have finite-dimensional unitary representations. (A familiar non-compact example is the Minkowski group.)

It turns out that there are only a finite set of classes of compact Lie groups. The *classical* groups are:

- $U(1)$ = phase transformations, $U = e^{i\alpha}$
- $SU(N)$ = set of $N \times N$ unitary matrices with $det(U) = 1$
- $SO(N)$ = set of orthogonal $N \times N$ matrices with $det(U) = 1$

- $Sp(2N)$ = set of $2N \times 2N$ matrices that preserve a slightly funny inner product.

In addition to these, there are the *exceptional* Lie groups: E_6, E_7, E_8, F_4, G_2. In studying the Standard Model, we will focus primarily on $U(1)$ and $SU(N)$ groups.

***e.g.* 2** $SU(2)$

This is the prototypical Lie group, and should already be familiar from what you know about spin in quantum mechanics. By definition, the corresponding Lie algebra has three basis elements which satisfy

$$[t^a, t^b] = i\epsilon^{abc} t^c \tag{15}$$

The basic *fundamental* representation of $SU(2)$ is in terms of Pauli matrices: $t^a = \sigma^a/2$. Since $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$, it's clear that this is a valid representation of the algebra. You might also recall that any $SU(2)$ matrix can be written in the form $U = \exp(i\alpha^a \sigma^a/2)$.

Some useful and fun facts about compact Lie algebras:

- Except for $U(1)$, we have $tr(t^a) = 0$ for all the classical and exceptional Lie groups.
- Number of generators = $d(G)$

$$d(G) = \begin{cases} N^2 - 1; & SU(N) \\ N(N-1)/2; & SO(N) \\ 2N(2N+1)/2; & Sp(2N) \end{cases} \tag{16}$$

- A representation (= rep) is *irreducible* if it cannot be decomposed into a set of smaller reps. This is true if and only if it is impossible to simultaneously block-diagonalize all the generators of the rep. Irreducible representation = irrep.
- If one of the generators commutes with all the others, it generates a $U(1)$ subgroup called an Abelian factor: $G = G' \times U(1)$.
- If the algebra cannot be split into sets of mutually commuting generators it is said to be *simple*. For example, $SU(5)$ is simple (as are all the classical and exceptional Lie groups given above) while $SU(3) \times SU(2) \times U(1)$ is not simple. In the latter case, all the $SU(3)$ generators commute with all the $SU(2)$ generators and so on.
- A group is *semi-simple* if it does not have any Abelian factors.
- With the basis choice yielding Eq. (14), one can show that the structure constants are completely anti-symmetric.
- The *fundamental* representation of $SU(N)$ is the set of $N \times N$ special unitary matrices acting on a complex vector space. This is often called the **N** representation. Similarly, the fundamental representation of $SO(N)$ is the set of $N \times N$ special orthogonal matrices acting on a real vector space.

- The *adjoint* ($= A$) representation can be defined in terms of the structure constants according to

$$(t^a_A)_{bc} = -if^{abc} \tag{17}$$

Note that on the left side, a labels the adjoint generator while b and c label its matrix indices.

- Given any rep t^a_r, the conjugate matrices $-(t^a_r)^*$ give another representation, unsurprisingly called the conjugate representation. A rep is said to be real if it unitarily equivalent to its conjugate. The adjoint rep is always real.

- The Casimir operator of a rep is defined by $T^2_r = t^a_r t^a_r$ (with an implicit sum on a). One can show that T^2_r commutes with all the t^a_r. For an irrep (=irreducible representation) of a simple group, this implies that

$$T^2_r = C_2(r)\mathbb{I}, \tag{18}$$

for some positive constant $C_2(r)$.

- It is conventional to fix the normalization of the fundamental of $SU(N)$ such that $T_2(\mathbf{N}) = 1/2$. Once this is done, it fixes the normalization of all the other irreps. In particular, it implies that for $SU(N)$, $C_2(\mathbf{N}) = (N^2-1)/2N$, $T_2(A) = N = C_2(A)$.

3 Gauge Invariance and QED

Recall the QED Lagrangian:

$$\mathscr{L} = \bar{\psi}i\gamma^\mu(\partial_\mu + ieQA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{19}$$

This theory clearly has a continuous symmetry for any fixed value of the parameter α:

$$\left\{ \begin{array}{ccc} \psi & \to & e^{iQ\alpha}\psi \\ A_\mu & \to & A_\mu \end{array} \right. \tag{20}$$

The corresponding symmetry group is $U(1)$. On the other hand, suppose we're feeling adventurous and decide to elevate the transformation parameter to a function on spacetime: $\alpha = \alpha(x)$. Doing so, we find that the transformation above is no longer a symmetry of the Lagrangian. In particular,

$$\bar{\psi}i\gamma^\mu\partial_\mu\psi \to \bar{\psi}i\gamma^\mu\partial_\mu\psi + \bar{\psi}i\gamma^\mu(iQ\partial_\mu\alpha)\psi. \tag{21}$$

Evidently the transformation of Eq. (20) is *not* a symmetry of the theory for non-constant parameters $\alpha(x)$.

We can restore the invariance of the fermion terms if we also have the vector field transform according to:

$$\begin{cases} \psi & \to & e^{iQ\alpha}\psi \\ A_\mu & \to & A_\mu - \frac{1}{e}\partial_\mu\alpha. \end{cases} \tag{22}$$

Together, this implies that

$$(\partial_\mu + ieQA_\mu)\psi \equiv D_\mu\psi \to e^{iQ\alpha}D_\mu\psi, \tag{23}$$

and therefore $\psi i\gamma^\mu D_\mu\psi$ is invariant under the transformation for arbitrary $\alpha(x)$ provided the vector field also transforms as indicated. The differential operator D_μ is sometimes called a *covariant derivative*. Even better, if we look at the effect of this shift on the photon kinetic term, we find that it remains unchanged as well:

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \to (\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{e}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\alpha = F_{\mu\nu} + 0. \tag{24}$$

Thus, QED is invariant under the transformations of Eq. (22) for any reasonable arbitrary function $\alpha(x)$.

At first glance this invariance might just seem like a clever trick, but the river beneath these still waters runs deep. Thinking back to regular electromagnetism (of which QED is just the quantized version), one often deals with scalar and vector potentials. These potentials are not unique and are therefore not observable (for the most part), and the true "physical" quantities are the electric and magnetic fields. The vector field A_μ in QED, corresponding to the photon, is identified with these potentials according to

$$A^\mu = (\phi, \vec{A}), \tag{25}$$

where ϕ and $\vec{A}$ are the usual scalar and vector potentials. This is justified by the equations of motion derived from the QED Lagrangian provided we also identify

$$F^{0i} = -E^i, \qquad F^{ij} = -\epsilon^{ijk}B^k, \tag{26}$$

with the electric and magnetic fields. With this identification, the transformations of Eq. (22) coincide with the usual "gauge" transformations you should have encountered in electromagnetism. Sometimes we call A_μ the gauge boson and the operation of Eq. (22) a gauge transformation.

Keeping in mind the story from electromagnetism, the interpretation of the quantum fields in QED is that ***only those quantities that are invariant under the transformations of Eq.*** (22) ***are physically observable.*** In particular, the vector field A_μ that represents the photon is not itself an observable quantity, but the gauge-invariant field strength $F_{\mu\nu}$ is. Put another way, the field variables we are using are redundant, and the transformations of Eq.(22) represent an *equivalence relation*: any two set of fields (ψ, A_μ) related by such a transformation represent the same physical configuration. Sometimes the invariance under Eq. (22) is called a local or gauge symmetry, but it is not really a symmetry at all. A true symmetry implies that different physical configurations have the same properties. Gauge invariance is instead a statement about which configurations are physically observable.

Gauge invariance is also sensible if we consider the independent polarization states of the photon, of which there are two. The vector field A_μ represents the photon, but it clearly has four independent components. Of these, the timelike polarization component is already non-dynamical on account of the form of the vector kinetic term. Invariance under gauge transformations effectively removes the additional longitudinal polarization leaving behind only the two physical transverse polarization states. Note as well that if the photon had a mass term, $\mathscr{L} \supset m^2 A_\mu A^\mu/2$, the theory would no longer be gauge invariant. Instead, the longitudinal polarization mode would enter as physical degree of freedom. Equivalently, gauge invariance forces the photon to be massless.

In the discussion above we started with the QED Lagrangian and showed that it was gauge-invariant. However, the modern view is to take gauge invariance as the fundamental principle. Indeed, the only way we know of to write a consistent, renormalizable theory of interacting vector fields is to have an underlying gauge symmetry. For QED, we could have started with a local $U(1)$ gauge invariance for a charged fermion field and built up the rest of the Lagrangian based on this requirement. In this context, the vector field is needed to allow us to define a sensible derivative operator on the fermion field, which involves taking a difference of two fields at different spacetime points with apparently different transformation properties, and corresponds to something called a *connection*.[4] Gauge invariance completely fixes the photon-fermion interactions, illustrating why it is so powerful. We will see shortly that gauge invariance is even more powerful when the underlying symmetry is a non-Abelian Lie group.

References

[1] H. Georgi, "Lie Algebras In Particle Physics. From Isospin To Unified Theories," Front. Phys. **54**, 1-255 (1982).

[2] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*

[3] L. Alvarez-Gaume, M. A. Vazquez-Mozo, "Introductory lectures on quantum field theory," [hep-th/0510040].

[4] R. Slansky, Phys. Rept. **79**, 1-128 (1981).

[4]See Ref. [2] for a nice explanation of these slightly cryptic comments.