PHYS 526 Notes #11: Path Integral Quantization

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In formulating quantum theories based on classical field theories, we have used the so-called *canonical quantization* procedure, where we elevated the classical Poisson brackets to (anti-)commutators between operators on a Hilbert space. It turns out there exists a second, equivalent way to formulate quantum theories called the *path integral* method. In some cases, and for theories with gauge invariance in particular, path integrals are much more convenient than canonical quantization. Understanding the path integral method also provides new insights on what we have already done.

You may or may not have encountered path integrals in your previous quantum mechanics courses. If not, don't worry since we will present everything you need to know here. Also, don't worry too much about mathematical exactitude. We don't really know how to define path integrals in a fully rigourous way. Nevertheless, they are extremely useful and they give a very nice intuitive picture of quantum mechanics. My favourite treatment of path integrals is Appendix A of Ref. [1], and much of these notes follows this text.[1]

1 Path Integrals for Scalar Fields

Let's begin with the easiest case, namely the scalar field $\phi(x)$. We will begin by defining path integrals as general mathematical objects. Next, we will show how they can be used to formulate a quantum theory of scalar fields.

1.1 Introduction to Path Integrals

Recall from `notes-02` that we defined the functional derivative with respect to a scalar field $\phi(x)$ according to

$$\frac{\delta f(\phi(x))}{\delta\phi(x')} = \frac{\partial f}{\partial\phi}\,\delta^{(4)}(x-x')\ . \tag{1}$$

This is just a continuous generalization of $\partial_\mu f(x) = \delta_\mu^{\ \nu}\,(\partial f/\partial x^\nu)$. Functional (or path) integrals will turn out to be an analogous generalization of regular integrals.

Let us first define the functional integral of the field $\phi(t,\vec{x})$ at the fixed time t_i. For this, we divide space into a lattice of points $\{\vec{x}_j\}$ labelled by $j = 1,\ldots M$. The functional integral at time t_i is then defined to be

$$\int[\mathscr{D}'\phi_i] \sim \lim_{M\to\infty}\prod_{j=1}^{M}\left[\int_{-\infty}^{\infty}d\phi(t_i,\vec{x}_j)\right]\ , \tag{2}$$

[1] You don't need to know any string theory to understand the appendix.

where the squiggle means equality up to an overall factor. The functional integral is therefore just the product of integrals over the field values at each point in space. As you can imagine, taking the *continuum limit* $M \to \infty$ is highly non-trivial, and not even necessarily well-defined.

To define the full path integral, let us take the time interval $[t', t'']$ and subdivide it into $(N+1)$ pieces. Take $t' = t_0$ and $t'' = t_{N+1}$, with t_i as the i-th intermediate time slice. The full path integral with fixed endpoints $\phi(t', \vec{x}) = \phi'(\vec{x})$ and $\phi(t'', \vec{x}) = \phi''(\vec{x})$ is then

$$\int [\mathscr{D}\phi]_{\phi'}^{\phi''} \sim \lim_{N\to\infty} \prod_{i=1}^{N} \int [\mathscr{D}'\phi_i] \ . \tag{3}$$

Note that the endpoints are fixed, and do not get integrated over. In many cases we will take $t' \to -\infty$, $t'' \to \infty$, and force ϕ', $\phi'' \to 0$. We will write the resulting path integral in this case as simply $\int [\mathscr{D}\phi]$. It is equivalent to the more symmetric expression

$$\int [\mathscr{D}\phi] \sim \prod_x \left[\int_{-\infty}^{\infty} d\phi(t, \vec{x})\right] \ . \tag{4}$$

You should think of this as a sum over all possible spacetime configurations of the field ϕ. The product over points in spacetime $x = (t, \vec{x})$ can also be viewed as the product over a lattice of points $x_i = (t_i, \vec{x}_i)$ in the limit that the lattice spacing is taken to zero.

The bad new about path integrals is that this definition is rather less than precise. However, the good news is that we only ever do a small number of different types of integrals. The first, and easiest, is

$$\int [\mathscr{D}\phi]\, \delta[\phi - \phi']\, S[\phi] = S[\phi'] \ , \tag{5}$$

where $S[\phi]$ is any functional of the fields and ϕ' is a specific function (*i.e.* a specific field configuration). The delta functional can also be written as a path integral, just like we have for the usual delta function. To derive it, note first that the delta functional is the product of delta functions at each point in spacetime, since both are zero unless the argument is the zero function:

$$\delta[\phi] \sim \prod_x \delta(\phi_x) \ , \tag{6}$$

where $\phi_x = \phi(x)$ is the value of the field at point x (and not a function of x). We also have

$$\delta(\phi_x) = \int \frac{d\omega_x}{2\pi}\, e^{i\omega_x\phi_x} \ . \tag{7}$$

It follows that

$$\delta[\phi] \sim \prod_x \left(\int d\omega_x\, e^{i\omega_x\phi_x}\right) \tag{8}$$

$$\sim \left(\prod_x \int d\omega_x\right) \exp(i \sum_x \omega_x\phi_x) \tag{9}$$

$$\sim \int [\mathscr{D}\omega] \exp\left[i \int d^4x\, \omega(x)\phi(x)\right] \ . \tag{10}$$

The second kind of integral we will encounter is the Gaussian. Recall that

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \ . \tag{11}$$

The path integral Gaussian is just a multi-dimensional generalization of this. To compute it, recall that the set of all functions on spacetime is an infinite-dimensional vector space with an inner product given by

$$f \cdot g = \int d^4x\, f(x) g(x) \ . \tag{12}$$

An element of the space is just a function f, and $f(x)$ is the x-th component of the function.[2]

The typical integrand of a functional Gaussian integral is

$$\exp\left[-\phi\cdot(\Delta\phi)\right] = \exp\left[-\int d^4x\ \phi(x)\Delta\,\phi(x)\right] \ , \tag{13}$$

where Δ is a differential operator with only even powers of ∂^2. Since we assume that ϕ vanishes at the boundary, this operator is Hermitian with respect to the inner product. It follows that we can expand any function ϕ in terms of a set of basis functions $\{f_A\}$ such that

$$\Delta\, f_A = \lambda_A f_A \ , \tag{14}$$

and

$$\int d^4x\ f_A f_B = \delta_{AB} \ . \tag{15}$$

Expanding ϕ in terms of these basis functions gives

$$\phi(x) = \sum_A \phi_A\, f_A(x) \ , \tag{16}$$

where ϕ_A is the A-th expansion coefficient. The Gaussian integrand then becomes

$$\exp\left[-\phi\cdot(\Delta\phi)\right] = \exp\left[-\sum_A \lambda_A \phi_A^2\right] = \prod_A e^{-\lambda_A\phi_A^2} \ . \tag{17}$$

Since any field configuration can be specified completely by the expansion coefficients ϕ_A, integrating over them is equivalent to doing the path integral. Thus, we have

$$\int [\mathscr{D}\phi] \sim \prod_A \left[\int_{-\infty}^{\infty} d\phi_A\right] \ . \tag{18}$$

[2]You've seen this in QM: if $|\psi\rangle$ is a state, $\psi(x) = \langle x|\psi\rangle$ is the x-th component in the position basis.

Putting these pieces together, we see that

$$\int [\mathscr{D}\phi]\, e^{-\phi\cdot(\Delta\phi)} \sim \prod_A \left[\int d\phi_A\, e^{-\lambda_A \phi_A^2}\right] \tag{19}$$

$$\sim \prod_A \left(\frac{\pi}{\lambda_A}\right)^{1/2} \tag{20}$$

$$\sim (\det\Delta)^{-1/2} . \tag{21}$$

Note that defining the determinant of an operator to be the product of its eigenvalues coincides with what we would find for a finite-dimensional diagonalizable matrix.

1.2 Path Integral Quantization of the Free Scalar

In `notes-02`, we discussed eigenstates of the Schrödinger-picture field operator at the fixed reference time $t = 0$, $\hat{\phi}(\vec{x}) = \hat{\phi}(0, \vec{x})$:[3]

$$\hat{\phi}(\vec{x})|\phi'\rangle = \phi'(\vec{x})|\phi'\rangle . \tag{22}$$

On the left-hand side we have the field operator (denoted by the hat), while on the right-hand side we have the specific classical field function that is its eigenvalue on the state $|phi'\rangle$. In the Heisenberg picture, we have

$$\hat{\phi}(x) = \hat{\phi}(t, \vec{x}) = e^{iHt}\hat{\phi}(0, \vec{x})e^{-iHt} . \tag{23}$$

In this picture, we can also form comoving eigenstates defined by

$$|\phi'(t)\rangle = e^{iHt}|\phi'\rangle . \tag{24}$$

These obviously satisfy

$$\hat{\phi}(t, \vec{x})|\phi'(t)\rangle = \phi'(\vec{x})|\phi'(t)\rangle . \tag{25}$$

Comoving eigenstates evolve oppositely to physical states in the Schrödinger picture.

The key result of path integral quantization is [1][4]

$$\langle\phi''(t'')|T\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\}|\phi'(t')\rangle = \int [\mathscr{D}\phi]_{\phi'}^{\phi''} \exp\left[i\int_{t'}^{t''} dt\, L(\phi, \dot{\phi})\right] \phi(x_1)\ldots\phi(x_n) , \tag{26}$$

where $t' < t_1, \ldots, t_n < t''$. This is a very deep relation. On the left side is a quantum mechanical matrix element of an operator. On the right, we just have a complicated sum over *classical* field configurations weighted by the action!

[3] In these notes, we will use a hat to denote a quantum operator.

[4] See Refs. [1, 2, 3, 4] for proofs.

For scattering calculations, we only need to find the expectation values of operators in the vacuum state $|\Omega\rangle$ of the theory. Similarly to what we did in `notes-03`, we can project it out by taking the limit of $t \to \pm\infty$ in a slightly imaginary direction. For example,

$$\lim_{t\to-\infty(1-i\epsilon)} |\phi'(t)\rangle = \lim_{t\to-\infty(1-i\epsilon)} e^{iHt'}\left(\sum_\lambda |\lambda\rangle\langle\lambda|\right)|\phi'\rangle \tag{27}$$

$$= \lim_{t\to-\infty(1-i\epsilon)} \left[|\Omega\rangle\langle\Omega|\phi'\rangle + \sum_{\lambda\neq\Omega} e^{iE_\lambda t}|\lambda\rangle\langle\lambda|\phi'\rangle\right] \tag{28}$$

$$= (\langle\Omega|\phi'\rangle)\,|\Omega\rangle \tag{29}$$

Instead of tilting t, we can instead add a small perturbation to the energy to make it slightly imaginary. A convenient way to do this in a field theory is to make the mass slightly imaginary, since the corresponding ϕ^2 operator is positive definite,

$$-\mathscr{L} \supset m^2\phi^2 \to (m^2 - i\epsilon)\phi^2 \ , \tag{30}$$

with $\epsilon > 0$. This induces

$$\mathscr{H}(m^2 - i\epsilon) \to \mathscr{H}(m^2) + \frac{\partial\mathscr{H}}{\partial m^2}(-i\epsilon) = \mathscr{H}(m^2) - i\epsilon' \ , \tag{31}$$

with $\epsilon' > 0$ since $\partial\mathscr{H}/\partial m^2 > 0$ in any sensible theory. This gives

$$E_\lambda \to E_\lambda - i\epsilon' \ , \tag{32}$$

which projects out the vacuum when inserted in Eq. (28).

After applying this projection, we obtain a master formula for path integrals:

$$\langle\Omega|T\{\hat\phi(x_1)\ldots\hat\phi(x_n)\}|\Omega\rangle = \int[\mathscr{D}\phi]\, e^{iS'[\phi]}\,\phi(x_1)\ldots\phi(x_n)\Big/\int[\mathscr{D}\phi]\, e^{iS'[\phi]} \ , \tag{33}$$

where the S' is the action with a slightly imaginary mass, and we have normalized the right-hand side so that $\langle 1\rangle = 1$. This formula should remind you of a statistical mechanical partition function, with $e^{iS'}$ instead of $e^{-H/T}$ as the exponential weight.

To study operator expectation values, let us define the *generating functional* $Z[J]$ by

$$Z[J] = \int[\mathscr{D}\phi]\,\exp(iS[\phi] + iJ\cdot\phi) = \int[\mathscr{D}\phi]\,\exp\left[i\int d^4x\,(\mathscr{L} + J\phi)\right] \ , \tag{34}$$

where $J(x)$ is an unspecified function and $J\cdot\phi = \int d^4x\, J(x)\phi(x)$ as before. The function $J(x)$ is often called the source. Suppose we take the functional integral of $Z[J]$ with respect to $J(x)$. This has the effect of adding a power of $i\phi(x)$ to the integrand:

$$\frac{\delta Z[J]}{\delta J(x)} = \int[\mathscr{D}\phi]\,\frac{\delta}{\delta J(x)} e^{i(S[\phi]+J\cdot\phi)} \tag{35}$$

$$= \int[\mathscr{D}\phi]\,\left[\int d^4y\, i\phi(y)\,\delta^{(4)}(y-x)\right]\, e^{i(S[\phi]+J\cdot\phi)} \tag{36}$$

$$= \int[\mathscr{D}\phi]\, i\phi(x)\, e^{i(S[\phi]+J\cdot\phi)} \tag{37}$$

Taking more derivatives would add more powers of ϕ to the integrand. We can use this to rewrite our master formula, Eq. (33), as

$$\langle\Omega|T\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\}|\Omega\rangle = (-i)^n \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\, Z[J]\Big|_{J=0} \Big/ \; Z[0] \; . \tag{38}$$

Setting $J \to 0$ after taking the derivatives gets rid of the $J\cdot\phi$ in the exponential, and normalizing by $1/Z[0]$ ensures that $\langle 1\rangle = 1$.

So far, all our results are completely general and apply to both free and interacting scalar theories. Let us now apply them specifically to the free scalar theory,

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(m^2 - i\epsilon)\phi^2 \; . \tag{39}$$

after integrating by parts, this gives

$$\begin{aligned} S[\phi] &= \frac{1}{2}\int d^4x\; \phi(-\partial^2 - m^2 + i\epsilon)\phi & (40)\\ &:= \frac{1}{2}\,\phi\cdot(\Delta\phi) \; , & (41)\end{aligned}$$

with $\Delta = (-\partial^2 - m^2 + i\epsilon)$. When inserted into the path integral, this will just be a Gaussian.

We can evaluate the generating functional for the free theory by completing the square and computing the Gaussian path integral. For this, note that

$$\begin{aligned} \Delta\; D_F(x) &= (-\partial^2 - m^2 + i\epsilon)\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 + i\epsilon}\,e^{-ip\cdot x} & (42)\\ &= i\delta^{(4)}(x) \; . & (43)\end{aligned}$$

We can therefore think of $-iD_F = \Delta^{-1}$. With this result in hand, we can complete the square by making the change of variable

$$\phi(x) = \phi'(x) - \int d^4y\,(-i)D_F(x-y)J(y) := \phi'(x) - (\Delta^{-1}J)(x) \; . \tag{44}$$

In the exponential of $Z[J]$, this gives

$$\begin{aligned} S[\phi] + J\cdot\phi &= \frac{1}{2}(\phi' - \Delta^{-1}J)\cdot[\Delta(\phi' - \Delta^{-1}J)] \;+\; J\cdot(\phi' - \Delta^{-1}J) & (45)\\ &= \frac{1}{2}\,\phi'\cdot(\Delta\phi') - \frac{1}{2}\,J\cdot(\Delta^{-1}J) & (46)\\ &= \frac{1}{2}\int d^4x\,\phi'\Delta\phi' + \frac{i}{2}\int d^4x\int d^4y\; J(x)D_F(x-y)J(y) \; . & (47)\end{aligned}$$

In going from the first to the second line we have implicitly integrated by parts to move the Δ operator around. The integrand in terms of ϕ' is now a Gaussian. Since the change of variables was just a shift by a constant, the path integral measure is also unchanged,

$$[\mathscr{D}\phi] = [\mathscr{D}\phi'] \; . \tag{48}$$

It follows that the free-theory generating functional is equal to

$$Z[J] \sim (\det \Delta)^{-1/2} \exp\left[-\frac{1}{2}\int d^4x \int d^4y \; J(x) D_F(x-y) J(y)\right] \; . \tag{49}$$

Note that the $i\epsilon$ factor inserted to project out the ground state leads to the Feynman propagator in this expression. It also ensures the convergence of the Gaussian integral.

We can now use Eq. (38) to compute n-point functions in the free theory. The 2-point functions comes out to be

$$(-i)^2 \, \frac{\delta^2}{\delta J(x)\, \delta J(y)} \left(e^{-\frac{1}{2} J\cdot D_F \cdot J}\right)\Big|_{J=0} = (-1)^2 D_F(x-y) \; , \tag{50}$$

which is precisely what we found previously. We also see that the propagator is just the inverse of the quadratic operator in the Lagrangian. Repeating with more derivatives, we reproduce all the n-point functions of the theory (and Wick's theorem) in a very trivial way. In this case, the contraction of two fields corresponds to the corresponding derivatives hitting the same term in the expansion of the exponential.

2 Path Integrals for Fermions

We can also quantize fermion fields using path integrals. For this, we need to define regular derivatives and integrals of Grassmann numbers. From here, it is straightforward to generalize to fermionic path integrals, and then to quantization.

2.1 Grassman Integrals, Regular and Path

Suppose we have a single Grassmann number η. Since $\eta^2 = 0$, the most general real function of η is

$$f(\eta) = a + b\eta \; , \tag{51}$$

for some real coefficients a and b. We define the derivative by

$$\frac{d}{d\eta} 1 = 0, \qquad \frac{d}{d\eta}\eta = 1 \; , \tag{52}$$

so that

$$\frac{d}{d\eta} f(\eta) = b \; . \tag{53}$$

For integrals, we define

$$\int d\eta \, 1 = 0, \qquad \int d\eta \, \eta = 1 \; . \tag{54}$$

It is best to think of $\int d\eta$ as an operator on functions of η rather than an integral in the usual sense. This definition implies that

$$\int d\eta \, \frac{df(\eta)}{d\eta} = 0 \; . \tag{55}$$

When there are multiple Grassmann numbers, we treat the integral and the derivative as operators acting from the left, and put everything together by anticommuting. For example, given η and χ we have

$$\frac{d}{d\eta}\chi = 0, \qquad \frac{d}{d\eta}(\chi\eta) = -\frac{d}{d\eta}(\eta\chi) = -\chi \; . \tag{56}$$

Integrals work in the same way.

Having warmed up with regular derivatives and integrals, let's generalize them to their functional counterparts. We have

$$\frac{\delta\Psi_a(x)}{\delta\Psi_b(y)} = \delta_a{}^b \, \delta^{(4)}(x-y) \; . \tag{57}$$

For the functional integral, we take

$$\int [\mathscr{D}\Psi] \sim \prod_x \left[\int d\Psi_1(x) \int d\Psi_2(x) \int d\Psi_3(x) \int d\Psi_4(x) \right] \; . \tag{58}$$

In other words, do the Grassmann integral over each component of Ψ at every point in spacetime and multiply together all the results. Note that $\Psi_a(x)$ should be treated as independent Grassmann variables in that $\Psi_a(x)\Psi_b(x')$ is non-zero unless $a = b$ and $x = x'$.

Evaluating fermionic functional integrals is even easier than for bosons. The case of most interest to us will the be the Gaussian,

$$\int [\mathscr{D}\Psi \, \mathscr{D}\bar{\Psi}] \; \exp\left[-\int d^4x \, \bar{\Psi}\Delta\Psi \right] \; , \tag{59}$$

where Δ is a Hermitian differential operator. As before, let us expand Ψ and $\bar{\Psi}$ in orthonormal eigenfunctions of the operator Δ:

$$\Psi(x) = \sum_A \psi_A g_A(x), \qquad \bar{\Psi}(x) = \sum_B \bar{\psi}_B h_B(x) \; , \tag{60}$$

where g_A and h_A are bosonic functions and ψ_A and $\bar{\psi}_B$ are Grassmannian expansion coefficients. For the eigenfunctions, we have

$$\Delta \, g_A = \lambda_A g_A \qquad \Delta \, h_B = \lambda_B h_B \; , \tag{61}$$

as well as

$$\int d^4x \, h_B(x) g_A(x) = \delta_{AB} \; . \tag{62}$$

Putting this into the integrand, we find

$$\exp\left[\bar{\Psi}\cdot(\Delta\Psi)\right] = \exp\left[-\sum_A \bar{\psi}_A\lambda_A\psi_A\right] = \prod_A e^{-\bar{\psi}_A\lambda_A\psi_A} \ . \tag{63}$$

The path integral measure becomes

$$\int[\mathscr{D}\Psi\,\mathscr{D}\bar{\Psi}] \sim \prod_A\left[\int d\psi_A\int d\bar{\psi}_A\right] \ . \tag{64}$$

The Gaussian integral is therefore equal to

$$\int[\mathscr{D}\Psi\,\mathscr{D}\bar{\Psi}]\,e^{-\bar{\Psi}\cdot(\Delta\Psi)} \sim \prod_A\left[\int d\psi_A\int d\bar{\psi}_A\,e^{-\bar{\psi}_A\lambda_A\psi_A}\right] \tag{65}$$

$$\sim \prod_A\left[\int d\psi_A\int d\bar{\psi}_A\,(1-\lambda_A\bar{\psi}_A\psi_A)\right] \tag{66}$$

$$\sim \prod_A(-\lambda_A) \tag{67}$$

$$\sim \det(\Delta) \ . \tag{68}$$

Note that the determinant has a positive power rather than a negative one. This is characteristic of fermions. The factor of +1 relative to 1/2 comes from the fact that two different fields are now involved.

2.2 Path Integral Quantization

With these tools in hand, we can now turn to physics. The master formula for a Dirac fermion is

$$\langle\Omega|T\{\hat{\Psi}_{a_1}(x_1)\ldots\hat{\bar{\Psi}}^{b_n}(x_n)\}|\Omega\rangle = \int[\mathscr{D}\Psi\,\mathscr{D}\bar{\Psi}]\,e^{iS'}\,\Psi_{a_1}(x_1)\ldots\bar{\Psi}^{b_n}(x_n)\Big/\int[\mathscr{D}\Psi\,\mathscr{D}\bar{\Psi}]\,e^{iS'}, \tag{69}$$

where $S' = S[\Psi,\bar{\Psi}] + i\epsilon\bar{\Psi}\Psi$. When there are multiple types of fields, we should integrate over each one of them in the path integral.

To compute n-point functions, it is again convenient to define a generating functional, this time with a source for each independent field. For a theory of Dirac fermions,

$$Z[\eta,\bar{\eta}] = \int[\mathscr{D}\Psi\mathscr{D}\bar{\Psi}]\,\exp\left[iS[\Psi,\bar{\Psi}] + i\bar{\eta}\cdot\Psi + i\bar{\Psi}\cdot\eta\right] \ . \tag{70}$$

We now have

$$\frac{\delta Z}{\delta\bar{\eta}^a(x)} = \int[\mathscr{D}\Psi\mathscr{D}\bar{\Psi}]\,(i\Psi_a(x))\,e^{iS[\Psi,\bar{\Psi}]+i\bar{\eta}\cdot\Psi+i\bar{\Psi}\cdot\eta} \ , \tag{71}$$

$$\frac{\delta Z}{\delta\eta_b(x)} = \int[\mathscr{D}\Psi\mathscr{D}\bar{\Psi}]\,(-i\bar{\Psi}^b(x))\,e^{iS[\Psi,\bar{\Psi}]+i\bar{\eta}\cdot\Psi+i\bar{\Psi}\cdot\eta} \ . \tag{72}$$

Note the extra minus sign from anticommuting the fermionic functional derivative. Taking multiple derivatives and setting $\eta = \bar{\eta} = 0$, we get

$$\langle\Omega|T\{\Psi_{a_1}(x_1)\ldots\bar{\Psi}^{b_n}(x_n)\}|\Omega\rangle = \frac{1}{Z[0,0]}\left(-i\frac{\delta}{\delta\bar{\eta}^{a_1}(x_1)}\right)\ldots\left(+i\frac{\delta}{\delta\eta_{b_n}(x_n)}\right) Z[\eta,\bar{\eta}]\bigg|_{\eta=\bar{\eta}=0} \quad (73)$$

Let's now specialize to the free theory of a Dirac fermion, with action

$$S[\Psi,\bar{\Psi}] = \int d^4x\;\bar{\Psi}(i\gamma\cdot\partial - m + i\epsilon)\Psi := \bar{\Psi}\cdot(\Delta\Psi)\;. \quad (74)$$

As before, we can complete the square in the generating functional and do the resulting Gaussian integral. The appropriate changes of variables in this case are

$$\Psi(x) \;=\; \Psi'(x) - \int d^4y\;(-i)S_F(x-y)\eta(y) \quad (75)$$

$$\bar{\Psi}(x) \;=\; \bar{\Psi}'(x) - \int d^4y\;(-i)\bar{\eta}(y)S_F(y-x)\;. \quad (76)$$

Using the fact that

$$\Delta[S_F(x)]_a^{\;b} = i\delta_a^{\;b}\delta^{(4)}(x)\;, \quad (77)$$

we obtain

$$Z[\eta,\bar{\eta}] \sim \det(\Delta)\;\exp\left[-\int d^4x\int d^4y\;\bar{\eta}(x)S_F(x-y)\eta(y)\right]\;. \quad (78)$$

As for the scalar, it is not hard to check that this formula reproduces all the n-point functions (and Wick's theorem) of the free Dirac theory.

3 Quantizing the Photon

Path integrals turn out to be very convenient for quantizing gauge theories. We will show how this works for the case of the free photon field.

3.1 Two Puzzles

The goal is to apply our path integral formalism to the free photon, with Lagrangian

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}A^\mu(-\eta^{\mu\nu}\partial^2 + \partial_\mu\partial_\nu)A^\nu\;, \quad (79)$$

where we have integrated by parts to get the second equality. Given this Lagrangian, we can construct an action and formulate a master formula for n-point functions in analogy with Eq. (33). The direct generalization is

$$\langle\Omega|T\{\hat{A}^{\mu_1}(x_1)\ldots\hat{A}^{\mu_n}(x_n)\}|\Omega\rangle \overset{?}{=} \int[\mathscr{D}A^\mu]\,A^{\mu_1}(x_1)\ldots A^{\mu_n}(x_n)e^{iS'[A]}\bigg/\int[\mathscr{D}A^\mu]\,e^{iS'[A]}\;, \quad (80)$$

where

$$\int [\mathscr{D}A^\mu] = \int [\mathscr{D}A^0] \ldots \int [\mathscr{D}A^3] \ . \tag{81}$$

It turns out that this direct generalization is not quite correct.

There are two immediate problems with Eq. (80). The first is that it involves an integration over many physically equivalent field configurations. In fact, given a configuration $A_\mu(x)$, there are infinitely many other configurations that give the same physics, namely

$$A_\mu^{'}(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \ , \tag{82}$$

for any function $\alpha(x)$. The integration in Eq. (80) therefore involves a huge (infinite!) overcounting.

The second problem with the attempt of Eq. (80) has to do with the propagator. Recall that for both the scalar and the fermion, we wrote the free theory as $\phi^{(*)} \cdot (\Delta\phi)$ for some Hermitian operator Δ. In the present case, we can do the same but with

$$\Delta \to \Delta_{\mu\nu} = \eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu \ . \tag{83}$$

If we were to follow our previous procedure and add a source and complete the square, we would obtain an operator contaning the inverse of $\Delta_{\mu\nu}$. Unfortunately, this inverse is not well-defined because $\Delta_{\mu\nu}$ has many zero eigenvalues. For example,

$$\Delta_{\mu\nu}(\partial^\nu\alpha) = 0 \ , \tag{84}$$

for any function α. Connecting with the first problem, we see that these zero eigenvalues correspond to field configurations that are gauge-equivalent to the zero configuration.

3.2 Gauge Fixing (Faddeev-Popov)

Both problems with the direct generalization of the path integral formula for scalars (and fermions) can be solved by taking into account the physics of gauge invariance. The idea will be to find a way to factor out the gauge redundancy and integrate over only physically distinct field configurations. It will turn out that this procedure also produces an invertible propagator factor.

As a first step, let us assume we can find a *gauge fixing* condition on the field A^μ that removes the gauge redundancy [5],

$$F(A_\mu) = 0 \ , \tag{85}$$

where the condition is to be applied at all spacetime points x. We will use the specific choice $F(A^\mu) = \partial_\mu A^\mu$ here, but the procedure presented here can be applied to other conditions. In general, a field configuration $\tilde{A}^\mu(x)$ will not satisfy the gauge fixing condition. However, in this Lorentz-gauge case there always exists a gauge-equivalent configuration A^μ for which the condition is satisfied, $A^\mu = \tilde{A}^\mu + \partial^\mu/e$. The gauge function α can be obtained by solving the equation

$$\partial^2\alpha = e(\partial_\mu A^\mu) \ . \tag{86}$$

In this way, the gauge fixing condition fixes the gauge function $\alpha(x)$, up to transformations with $\partial^2\alpha = 0$. A good gauge fixing condition is one that always has a solution, with that solution completely fixing the gauge.

Given the field configuration A^μ, let us denote the gauge-transformed version by

$$A^\mu_\alpha(x) = A^\mu(x) + \frac{1}{e}\partial^\mu\alpha(x) \tag{87}$$

Next, let us also define the functional $\Delta_{FP}[A]$ by

$$1 = \Delta_{FP}[A^\mu]\int[\mathscr{D}\alpha]\,\delta[F(A^\mu_\alpha)] \ . \tag{88}$$

As long as the gauge condition has a solution, the delta functional will give a non-zero result for any configuration A^μ. The functional $\Delta_{FP}[A]$ is then just the inverse of the number left over. A key property of $\Delta_{FP}[A]$ is that it is gauge invariant, $\Delta_{FP}[A^\mu_{\alpha'}] = \Delta_{FP}[A^\mu]$ for any gauge transformation function $\alpha'(x)$. To see this, just plug the transformed result into the transformation and use the fact that successive transformations by α' and α are equivalent to a single transformation by $\alpha'' = (\alpha+\alpha')$. Explicitly,

$$\begin{aligned}
\Delta^{-1}_{FP}[A^\mu_{\alpha'}] &= \int[\mathscr{D}\alpha]\,\delta[F(A^\mu_{\alpha'+\alpha})] && (89)\\
&= \int[\mathscr{D}\alpha'']\,\delta[F(A^\mu_{\alpha''})] && (90)\\
&= \Delta^{-1}_{FP}[A^\mu] \ . && (91)
\end{aligned}$$

In the second line we have used $[\mathscr{D}\alpha''] = [\mathscr{D}\alpha]$, since the two differ only by a shift by a fixed function $\alpha'(x)$.

Let's take this result and plug it into the naïve path integral expression we attempted to use in Eq. (80),

$$\int[\mathscr{D}A^\mu]\,\mathcal{O}(A^\mu)\,e^{iS'[A^\mu]} = \int[\mathscr{D}A^\mu]\left(\Delta_{FP}[A^\mu]\int[\mathscr{D}\alpha]\,\delta[F(A^\mu_\alpha)]\right)\mathcal{O}(A^\mu)\,e^{iS[A^\mu]} \ , \tag{92}$$

where $\mathcal{O}(A^\mu)$ is a gauge-invariant time-ordered operator built out of A^μ fields, and we have just inserted unity in the form of Eq. (88). We can reorganize this expression by interchanging the orders of the functional integrals, and using gauge invariance in the form $\Delta_{FP}(A^\mu_\alpha) = \Delta_{FP}(A^\mu)$, $S[A^\mu_\alpha] = S[A^\mu]$, and $\mathcal{O}(A^\mu_\alpha) = \mathcal{O}(A^\mu)$. This gives

$$\begin{aligned}
\int[\mathscr{D}A^\mu]\,\mathcal{O}(A^\mu)\,e^{iS'[A^\mu]} &= \int[\mathscr{D}\alpha]\int[\mathscr{D}A^\mu_\alpha]\,\Delta_{FP}[A^\mu_\alpha]\,\delta[F(A^\mu_\alpha)]\,\mathcal{O}(A^\mu_\alpha)\,e^{iS[A^\mu_\alpha]} && (93)\\
&= \left(\int[\mathscr{D}\alpha]\right)\int[\mathscr{D}A^\mu]\,\Delta_{FP}[A^\mu]\,\delta[F(A^\mu)]\,\mathcal{O}(A^\mu)\,e^{iS[A^\mu]} \ , && (94)
\end{aligned}$$

where in the last line we have used $[\mathscr{D}A^\mu_\alpha] = [\mathscr{D}A^\mu]$, since the integration variables only differ by a constant shift.

The result of Eq. (94) suggests how to build a reasonable quantum theory of the photon. In this expression, we have reorganized the simple path integral over all gauge field configurations into an expression that is independent of gauge times a functional integral over all gauge transformation parameters. This gauge factor is sometimes called the *volume* of the gauge group, and is formally infinite. It is also precisely the factor corresponding to the number of times we are overcounting physically equivalent field configurations. The sensible thing to do, therefore, is cancel it off and reinterpret the remaining factor as the correct expression for the time ordered, gauge-invariant operator matrix element:

$$\langle\Omega|T\{\mathcal{O}(A^\mu)\}|\Omega\rangle = \int[\mathscr{D}A^\mu]\,\Delta_{FP}[A^\mu]\,\delta[F(A^\mu)]\,\mathcal{O}(\mathcal{A}^\mu)\,e^{iS[A^\mu]} \tag{95}$$

$$\Big/ \int[\mathscr{D}A^\mu]\,\Delta_{FP}[A^\mu]\,\delta[F(A^\mu)]\,e^{iS[A^\mu]}\;. \tag{96}$$

This expression looks just like Eq. (80), except that we now have an additional factor of $\Delta_{FP}[A^\mu]\delta[F(A^\mu)]$ in both the numerator and denominator.

To evaluate Eq. (95) for a given operator, we need to figure out how to handle $\Delta_{FP}[A^\mu]$, defined by Eq. (88). This will require some more functional funny business. Before addressing that, let us start of with a regular multi-dimensional integral. Recall that here, changing integration variables generates a Jacobian factor. For example, consider $(x^1, x^2) \to (u^1, u^2)$, and the integral

$$\int d^2x\,\delta^{(2)}(u-u_0) = \int d^2u\,\left[\det\left(\frac{\partial u^i}{\partial x^j}\right)\right]^{-1}\delta^{(2)}(u-u_0) = \left[\det\left(\frac{\partial u^i}{\partial x^j}\right)\right]^{-1}_{u=u_0}\;. \tag{97}$$

The remaining determinant is the so-called Jacobian, and it is to be evaluated at points where $u = u_0$.

We would like to do the same type of change of variables for the functional integral relation of Eq. (88). First, note that $F(A^\mu_\alpha)$ should be set to zero at each point x, and to keep track of this, we will write $F(x, A^\mu_\alpha)$. The change of variables will therefore be $\alpha(y) \to F(x, A^\mu_\alpha)$. In direct analogy with the regular multi-dimensional integral, this change of variables produces a Jacobian factor

$$\Delta^{-1}_{FP}(A^\mu) = \int[\mathscr{D}\alpha]\,\delta[F(x, A^\mu_\alpha)] = \left|\det\left[\frac{\delta F(x, A^\mu_\alpha)}{\delta\alpha(y)}\right]\right|^{-1}_{F(A_\alpha)=0}\;. \tag{98}$$

This looks like a mess, but we can evaluate it for the specific case of $\partial_\mu A^\mu_\alpha(x) = 0$. Using gauge invariance, we can choose our representative A^μ such that it satisfies $F(x, A^\mu) = 0$ with $\alpha(x) = 0$. This implies that

$$\Delta_{FP}(A^\mu) = \left|\det\left[\frac{\delta F(x, A^\mu_\alpha)}{\delta\alpha(y)}\right]\right| = \left|\det\left(\frac{1}{e}\,\partial^2_x\,\delta^{(4)}(x-y)\right)\right|\;. \tag{99}$$

Note that this is independent of A^μ. We can rewrite this determinant in terms of a fermionic functional integral,

$$\Delta_{FP} \sim \int [\mathscr{D}c][\mathscr{D}\bar{c}] \, \exp\left[\frac{i}{e}\int d^4x \, \bar{c}\partial^2 c\right] \; . \tag{100}$$

The fields c and $\bar{c}$ are fermions called *Faddeev-Popov ghosts*. Since they don't couple to anything and don't appear as physical external particles, we can ignore them for QED. However, in more complicated gauge theories they couple to the gauge field and must be included in calculations at loop level.

As a final trick, let us rewrite the $\delta[F(A^\mu)]$ delta functionals in a nicer form. To do so, we modify the gauge condition to $F(A^\mu_\alpha) - \omega(x) = 0$ for some fixed function ω. All our previous work goes through in exactly the same way, with the only change being the replacement $\delta[F(A^\mu)] \to \delta[F(A^\mu) - \omega]$. Next, let us sum over many different functions $\omega(x)$ and multiply by a weighting functional $G[\omega]$ of the form

$$G[\omega] = \exp\left[\frac{-i}{2\xi}\int d^4x \, \omega^2(x)\right] \; . \tag{101}$$

At the end of the day, this gives (in the numerator of Eq. (95))

$$\int [\mathscr{D}A^\mu] \int [\mathscr{D}\omega] \, \delta[F(A^\mu) - \omega] \, G[\omega] \, \Delta_{FP} \, \mathcal{O}(A^\mu) \, e^{iS[A^\mu]} \tag{102}$$

$$= \int [\mathscr{D}A^\mu] \int [\mathscr{D}\omega] \, G[F(A^\mu)] \, \Delta_{FP} \, \mathcal{O}(A^\mu) \, e^{iS[A^\mu]} \tag{103}$$

$$= \int [\mathscr{D}A^\mu] \int [\mathscr{D}c][\mathscr{D}\bar{c}] \; \mathcal{O}(A^\mu) \, e^{i(S+S_{gf}+S_c)} \; , \tag{104}$$

where

$$S_{tot} = S + S_{gf} + S_c = \int d^4x \, \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{e}\bar{c}\partial^2 c\right] \; . \tag{105}$$

Note that we have used the delta functional to remove the integration over ω and replace it everywhere by $F(A^\mu)$.

We have just done a lot of functional heavy lifting, but the final result is very simple. The procedure of cancelling off the gauge redundancy of the functional integral amounts to adding fermionic ghost fields and some new terms in the action. In summary, removing the gauge redundancy of the functional integral can be reduced simply modifying the naïve formula of Eq. (80) by:

1. Include Dirac fermion ghost fields c and $\bar{c}$ with action S_c. The fields do not couple to anything in QED and can be neglected. (This is not true in more complicated theories like QCD!)

2. Add a *gauge-fixing term* -$(\partial_\mu A^\mu)^2/2\xi$ to the action. This is the same form as the extra term we added in Gupta-Bleuler quantization.

That's it!

3.3 Propagation

Our complicated gauge-fixing procedure has removed the overcounting of configurations in the functional integral. Let us next address the second problem of a non-invertible photon quadratic term. It turns out that gauge fixing solves this problem too by adding a new piece to the action. After integrating by parts, we now have

$$S_{tot} = \int d^4x\, \frac{1}{2} A^\mu \left[\eta_{\mu\nu}\partial^2 - (1-1/\xi)\partial_\mu\partial_\nu\right] A^\nu \ . \tag{106}$$

This is invertible for any finite ξ. Introducing a source and completing the square as before, we obtain

$$\langle\Omega|T\{A^\mu(x)A^\nu(x')\}|\Omega\rangle = \int d^4k\; e^{-ik\cdot(x-x')}\frac{i}{k^2+i\epsilon}\left[-\eta^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right] \ . \tag{107}$$

Again, this matches what we had in `notes-10`.

4 Interacting Theories

We have already used path integrals to reproduce all the operator expectation values of the free scalar, fermion, and photon theories studied previously. In this section we show how to use path integrals to compute operator expectation values in interacting theories.

To start, consider the interacting scalar theory given by

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 & (108)\\ &= \mathscr{L}_0 - \Delta V(\phi) \ . & (109) \end{aligned}$$

Let us also write the generating functional of the free theory as $Z_0[J]$,

$$Z_0[J] := \int [\mathscr{D}\phi] e^{i(S_o + J\cdot\phi)} \sim e^{-\frac{1}{2}J\cdot D_F \cdot J} \ , \tag{110}$$

where $S_0 = \int d^4x\, \mathscr{L}_0$. This is a nice result, but what we really want is the full generating functional $Z[J]$ so that we can compute n-point functions using Eq. (38).

In general, we do not know how to compute the full functional $Z[J]$ exactly. However, it is possible to derive a simple relation between $Z[J]$ and $Z_0[J]$ that lends itself to perturbation theory. For this, note that

$$\int d^4x\, \Delta V\left(-i\frac{\delta}{\delta\phi(x)}\right) Z_0[J] = \int [\mathscr{D}\phi] \left(\int d^4x\, \frac{\lambda}{4!}\phi^4(x)\right) e^{i(S_0 + iJ\cdot\phi)} \ , \tag{111}$$

where $V\left(-i\frac{\delta}{\delta\phi(x)}\right)$ means to replace every appearance of $\phi(x)$ in ΔV with $\delta/\delta\phi(x)$. Thus,

$$Z[J] = \exp\left[-i\int d^4x\, \Delta V\left(-i\frac{\delta}{\delta\phi(x)}\right)\right] Z_0[J] \ . \tag{112}$$

This together with Eq. (38) are the path integral form of the master formula from `notes-03`. Computing n-point functions is now just a matter of taking lots of derivatives.

To illustrate this procedure in a more complicated example, let us compute $\langle\phi(x_1)\Psi_a(x_2)\bar{\Psi}^b(c_3)\rangle$ in QED Lite. Introducing sources for the scalar and fermion fields as in the free theory, we have

$$Z[J,\eta,\bar{\eta}] = \exp\left[-iy\int d^4z\left(-i\frac{\delta}{\delta J(z)}\right)\left(+i\frac{\delta}{\delta\eta_c(z)}\right)\left(-i\frac{\delta}{\delta\bar{\eta}^c(z)}\right)\right] Z_0[J,\eta,\bar{\eta}]\ , \quad (113)$$

where

$$Z_0[J,\eta,\bar{\eta}] = Z_0[0,0,0]\, e^{-\frac{1}{2}J\cdot D_F\cdot J}\ e^{-\bar{\eta}\cdot S_F\cdot\eta}\ . \quad (114)$$

To get the 3-point function, we should differentiate with respect to the corresponding sources at x_1, x_2, and x_3. Thus,

$$\langle\phi(x_1)\Psi_a(x_2)\bar{\Psi}^b(c_3)\rangle = \left(-i\frac{\delta}{\delta J_1}\right)\left(i\frac{\delta}{\delta\eta_{a2}}\right)\left(-i\frac{\delta}{\delta\bar{\eta}^b_3}\right) Z[J,\eta,\bar{\eta}]\big|_{J=0=\eta=\bar{\eta}} \Big/ Z[0,0,0]\ . \quad (115)$$

To evaluate this, we expand the exponential in Eq. (113). At leading order in the coupling y, we have

$$\langle\phi(x_1)\Psi_a(x_2)\bar{\Psi}^b(c_3)\rangle \quad (116)$$

$$= (iy)\frac{\delta}{\delta J_1}\frac{\delta}{\delta\eta_{a2}}\frac{\delta}{\delta\bar{\eta}^b_3}\int d^4z\,\frac{\delta}{\delta J(z)}\frac{\delta}{\delta\eta_c(z)}\frac{\delta}{\delta\bar{\eta}^c(z)}e^{-\frac{1}{2}J\cdot D_F\cdot J}\ e^{-\bar{\eta}\cdot S_F\cdot\eta}\ |_{J=0=\eta=\bar{\eta}} \quad (117)$$

$$= (-iy)\int d^4z\, D_F(x_1-z)\,[S_F(x_2-z)]_a^{\ c}\,[S_F(z-x_3)]_c^{\ b} + \text{(disconnected)} \quad (118)$$

This matches what we found in `notes-08`.

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