PHYS 526 Notes #10: Quantizing the Photon Field

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Our remaining task is to justify the photon propagator and external state sums we presented in the Feynman rules for QED. In doing so, we will also gain some intuition about how gauge invariance manifests itself within Feynman diagrams.

1 Classical Vector Fields

The classical Lagrangian of electromagnetism (in the absence of sources) is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2}$$

Under Lorentz transformations, the vector field A^μ transforms as

$$A^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \ , \tag{3}$$

corresponding to the 4-vector representation.

Everything looks good so far, but a puzzle arises when we try to quantize the theory. We know that the photon has two independent polarization states, corresponding to the two possible helicities of a massless particle. However, the vector field A^μ has four components, and it would seem that the theory should have four particle states. The solution to this puzzle will turn out to be gauge invariance.

1.1 Degrees of Freedom: Massive Vector

Before tackling the photon, let us begin with the slightly easier case of a massive vector field Z^μ. Even though this field would seem to have four degrees of freedom, we can use it to build a quantum theory of a massive particle with spin $s = 1$, with three degrees of freedom. To see how, note that under the rotation subgroup of Lorentz a general 4-vector decomposes into states with $s = 0$ and $s = 1$. The $s = 0$ piece corresponds to the subset of Z^μ fields that can be written as $Z_\mu = \partial_\mu\phi$ for some scalar ϕ. To project these configurations out to isolate the $s = 1$ part, it turns out to be sufficient to apply the constraint $\partial_\mu Z^\mu = 0$.

To see how this works, let us try using a Lagrangian for Z^μ that is a simple generalization of electromagnetism,

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{1}{2}m^2 Z_\mu Z^\mu & (4)\\ &= \frac{1}{2}Z^\mu(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)Z^\nu + \frac{1}{2}\,m^2 Z_\mu Z^\mu \ , & (5)\end{aligned}$$

where $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$, and we have integrated by parts in the second line and dropped the total derivative that will vanish when inserted into the action. The only new thing here relative to electromagnetism is the mass term.

The equations of motion implied by this Lagrangian are

$$0 = (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)Z^\nu + m^2 Z_\mu \tag{6}$$

Applying ∂^μ to this result, we find the condition

$$m^2 \partial_\mu Z^\mu = 0 \ . \tag{7}$$

Thus, the only way to have a non-trivial solution is for the condition $\partial_\mu Z^\mu = 0$ to be satisfied. This is also consistent with the absence of a $s = 0$ mode. Consider what happens in this Lagrangian when $Z_\mu = \partial_\mu \phi$. The kinetic term gives

$$Z_{\mu\nu} \to \partial_\mu(\partial_\nu\phi) - \partial_\nu(\partial_\mu\phi) = 0 \ . \tag{8}$$

In the mass term, we find

$$m^2 Z_\mu Z^\mu \to m^2 \partial_\mu\phi\, \partial^\mu\phi \to -m^2\phi\, \partial^2\phi \ . \tag{9}$$

This vanishes as well if we impose the constraint

$$0 = \partial_\mu Z^\mu \to \partial_\mu(\partial^\mu\phi) \ . \tag{10}$$

Thus, the funny kinetic term plus the constraint imply that the action does not depend at all on the $s = 0$ part of Z^μ.

It is also possible to couple the massive vector to other stuff without reintroducing a dependence on the $s = 0$ part. If we were to write an interaction for Z^μ that is linear in the field, it would have to take the form

$$\mathscr{L} \supset Z_\mu j^\mu \ , \tag{11}$$

for some four-vector operator j^μ. Putting $Z_\mu = \partial_\mu\phi$ into Eq. (11), we find

$$\mathscr{L} \supset \partial_\mu\phi\, j^\mu = -\phi\, (\partial_\mu j^\mu) \ , \tag{12}$$

up to a total derivative. This implies that the action will not depend on the $s = 0$ component of Z^μ if and only if the operator j^μ is a conserved current, $\partial_\mu j^\mu = 0$.

With the constraint, the equations of motion for the free massive vector become

$$0 = (\partial^2 + m^2)Z_\mu - \partial_\mu(\partial_\nu Z^\nu) \ , \tag{13}$$

which is just a Klein-Gordon equation for each of the components. A general solution is

$$Z_\mu = \sum_{\lambda=1}^{3} \int \widetilde{dk}\ \left[a(k,\lambda)\epsilon_\mu(k,\lambda)e^{-ik\cdot x} + a^*(k,\lambda)\epsilon^*_\mu(k,\lambda)e^{ik\cdot x}\right] \ , \tag{14}$$

where the *polarization* 4-vectors ϵ^μ must satisfy

$$k^\mu \epsilon_\mu(k,\lambda) = 0 \ . \tag{15}$$

This condition comes from the $\partial_\mu Z^\mu = 0$ constraint. Three of them are needed to make up a basis of 4-vectors subject to the one constraint. For $k^\mu = (E_k, 0, 0, k)$, a convenient choice for them is

$$\epsilon^\mu(1) = (0,1,0,0), \quad \epsilon^\mu(2) = (0,0,1,0), \quad \epsilon^\mu(3) = \left(\frac{k}{m}, 0, 0, \frac{E_k}{m}\right) \ . \tag{16}$$

The first two are called transverse polarizations, while the third is said to be longitudinal.

1.2 Degrees of Freedom: Massless Vector

We turn next to the free photon. The Lagrangian can be rewritten as

$$\mathscr{L} \ = \ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}A^\mu(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu \ , \tag{17}$$

where we have again dropped a total derivative. To relate this to a massless particle with two helicity states is more difficult than the massive case, and is not just a matter of taking the mass to zero.

The key new feature of the massless vector Lagrangian of Eq. (17) is the invariance under gauge transformations

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha \ , \tag{18}$$

for any smooth function α. As discussed in `notes-09`, there is also a new physical interpretation of the theory to go along with this invariance. Specially, field configurations related by a gauge transformation are understood to be physically equivalent. In the language that is typically used in classical electromagnetism, the physical quantities are the electric and magnetic fields, not the scalar and vector potentials, $A^\mu = (\phi, \vec{A})$.

The equations of motion implied by Eq. (17) are

$$0 \ = \ \partial^2 A^0 - \partial_0(\partial_0 A^0 + \partial_i A^i) \ = \ -\vec{\nabla}^2 A^0 - \partial_0(\vec{\nabla}\cdot\vec{A}) \tag{19}$$

$$0 \ = \ -\partial^2 A^i - \partial_i(\partial_0 A^0 + \partial_j A^j) \ , \tag{20}$$

where we have written the time and space components separately. Note that Eq. (19) is just a Poisson equation for A^0. It implies that we can solve for A^0 in terms of the spatial components [1],

$$A^0(\vec{x}) = \frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}\cdot(\partial\vec{A}/\partial t)}{|\vec{x}-\vec{x}'|} \ . \tag{21}$$

This looks promising for the quantum theory, since it looks like we will be able to remove A^0 as a dynamical variable.

To actually solve the equations of motion for A^μ, it helps enormously to choose a specific gauge. A popular example is the *Coulomb gauge*,

$$\vec{\nabla}\cdot\vec{A} = 0\ . \tag{22}$$

In this case, we get $A^0 = 0$ together with $\partial^2\vec{A} = \vec{0}$. The general solution is

$$A^i(x) = \sum_{\lambda=1,2}\int\widetilde{dk}\left[a(\vec{k},\lambda)\epsilon^i(\vec{k},\lambda)e^{-ik\cdot x} + a^*(\vec{k},\lambda)\epsilon^{i\,*}(\vec{k},\lambda)e^{ik\cdot x}\right]\ , \tag{23}$$

with $k^0 = \sqrt{\vec{k}^2}$. Applying the Coulomb gauge condition, we must have $\vec{k}\cdot\vec{\epsilon} = 0$. A convenient choice of basis 3-vectors for $\vec{k} = (0,0,k)$ are the linear polarizations

$$\epsilon^i(1) = (1,0,0), \quad \epsilon^i(2) = (0,1,0)\ . \tag{24}$$

A second popular choice are the right- and left-handed circular polarizations

$$\epsilon^i(1) = \frac{1}{\sqrt{2}}(1,i,0), \quad \epsilon^i(2) = \frac{1}{\sqrt{2}}(1,-i,0)\ . \tag{25}$$

In this gauge, we can interpret $\vec{A}$ as a vector wave propagating at the speed of light with two independent polarizations. The downside of the Coulomb gauge is that it obscures the underlying Lorentz invariance of the theory.

A second gauge choice is the Lorentz-invariant *Lorenz gauge*,[1]

$$\partial_\mu A^\mu = 0\ . \tag{26}$$

A convenient way to impose this condition on the classical theory is to use a Lagrange multiplier ξ. This means we modify the Lagrangian to

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\ , \tag{27}$$

while treating the A^μ fields as being unconstrained but also promoting ξ to a variable of the system. Without having to worry about the constraint, it is easy to derive the equations of motion for A^μ,

$$\left[-\eta_{\mu\nu}\partial^2 + (1-1/\xi)\partial_\mu\partial_\nu\right]A^\nu = 0\ . \tag{28}$$

However, we also have to include the equation of motion for ξ, which yields $\partial_\mu A^\mu = 0$ and reproduces the constraint.

2 Quantizing the Vector

We turn next to quantizing the massless vector. This will turn out to be fairly complicated, with a number of subtleties due to gauge invariance and the related fact that we have more field variables than physical states. We will also work specifically within the Lorenz gauge as in Refs. [1, 2]. See Srednicki [3] for a discussion of photon quantization in the Coulomb gauge.

[1] Often called Lorentz gauge, because the corresponding condition is Lorentz invariant. Poor Lorenz.

2.1 Hamiltonian Formulation

As usual, the first step is to find the classical Hamiltonian. For now, we will not worry about the gauge choice $\partial_\mu A^\mu = 0$, and just work with Eq. (27) as the defining Lagrangian. We will also fix $\xi = 1$. This isn't necessary, but it greatly simplifies the algebra.

The conjugate momenta to the A^μ are

$$\Pi_\mu = \frac{\partial \mathscr{L}}{\partial(\partial_0 A^\mu)} = -F_{0\mu} \ . \tag{29}$$

The Hamiltonian is therefore

$$\begin{aligned} H &= \int d^3x \, [\Pi_\mu \partial_0 A^\mu - \mathscr{L}] && (30) \\ &= \int d^3x \left(\frac{1}{2}\left[(\pi^i)^2 + (\vec{\nabla} A^i)^2\right] - \frac{1}{2}\left[(\pi^0)^2 + (\vec{\nabla} A^0)^2\right] \right) && (31) \end{aligned}$$

The negative sign in the A^0 piece looks strange, but let's keep on trucking.

2.2 Going Quantum

The next step is to elevate the field variables and their conjugate momenta to operators on a Hilbert space. The equal-time commutation relations are

$$\begin{aligned} [A^\mu(t,\vec{x}),\, \Pi_\nu(t,\vec{x}')] &= i\,\delta^\mu{}_\nu\, \delta^{(3)}(\vec{x}-\vec{x}') \ , && (32) \\ [A^\mu(t,\vec{x}),\, A^\nu(t,\vec{x}')] &= 0 = [\Pi_\mu(t,\vec{x}), \Pi_\nu(t,\vec{x}')] \ . && (33) \end{aligned}$$

Note that each of the components of A^μ is treated as an independent variable.

Following the same procedure as for the scalar, we can rewrite the vector field in terms of mode operators,

$$A^\mu(x) = \int \widetilde{dk} \sum_{\lambda=0}^{3} \left[a(k,\lambda)\epsilon^\mu(k,\lambda) e^{-ik\cdot x} + a^\dagger(k,\lambda)\epsilon^{\mu *}(k,\lambda) e^{ik\cdot x} \right] \ , \tag{34}$$

where $k^0 = |\vec{k}|$, and the four $\epsilon^\mu(k,\lambda)$ form a basis of four vectors. It is convenient (and always possible) to choose these 4-vectors such that [2]

$$\begin{aligned} k\cdot\epsilon(k,1) &= k\cdot\epsilon(k,2) = 0 && (35) \\ k\cdot\epsilon(k,0) &= -k\cdot\epsilon(k,3) && (36) \\ \epsilon^\mu(k,\lambda)\epsilon_\mu(k,\lambda') &= \eta^{\lambda\lambda'} && (37) \end{aligned}$$

With this choice, the commutation relations of the mode operators implied by the ETCRs are

$$\begin{aligned} [a_{k,\lambda},\, a^\dagger_{p,\lambda'}] &= -\eta^{\lambda\lambda'}(2\pi)^3\, 2k^0\, \delta^{(3)}(\vec{k}-\vec{p}) && (38) \\ [a_{k,\lambda},\, a_{p,\lambda'}] &= 0 = [a^\dagger_{k,\lambda},\, a^\dagger_{p,\lambda'}] && (39) \end{aligned}$$

[2]For example, if $k^\mu = (k,0,0,k)$, $\epsilon(0) = (1,0,0,0)$, $\epsilon(1) = (0,1,0,0)$, $\epsilon(2) = (0,0,1,0)$, $\epsilon(3) = (0,0,0,1)$.

In terms of these modes operators, the Hamiltonian is given by

$$
\begin{aligned}
H &= \int \widetilde{dk}\, k^0 \sum_{\lambda\lambda'} (-\eta^{\lambda\lambda'}) a^\dagger_{k,\lambda} a_{k,\lambda'} && (40) \\
&= \int \widetilde{dk}\, k^0 \left[\sum_{\lambda=1}^{3} a^\dagger_{k,\lambda} a_{k,\lambda} - a^\dagger_{k,0} a_{k,0} \right] . && (41)
\end{aligned}
$$

To build the Hilbert space, we assume there exists a vacuum state $|0\rangle$ such that

$$
a_{k,\lambda}|0\rangle = 0 . \tag{42}
$$

All other states in the space can be built by applying powers of $a^\dagger_{k,\lambda}$ to $|0\rangle$. We interpret these states as free photons with four-momentum k^μ and polarization λ.

2.3 Problems with this Quantum Theory

As it stands, the quantum theory we've just developed has a number of puzzling features. We collect here a list of the worst of them. In the next subsection, we will present a way to fix them.

i) One-Particle States

This theory has four one-particle states, corresponding to $|k, \lambda\rangle = a^\dagger_{k,\lambda}|0\rangle$ for $\lambda = 0, 1, 2, 3$. However, we know that a real photon has only two independent polarizations. This isn't too surprising because we haven't used gauge invariance yet.

ii) Commutators

The commutator of Eq. (38) looks just like what we would expect for independent scalar fields for $\lambda = 1, 2, 3$, but it has the opposite sign for $\lambda = \lambda' = 0$.

iii) Inner Products

Consider the inner product of a pair of one-particle states,

$$
\langle k, \lambda | p, \lambda' \rangle = -\eta^{\lambda\lambda'} (2\pi)^3\, 2k^0\, \delta^{(3)}(\vec{k} - \vec{p}) . \tag{43}
$$

Again, this is fine for $\lambda = 1, 2, 3$, but it is negative for $\lambda = \lambda' = 0$. The inner product therefore fails to be positive definite. This same problem can arise for multi-particle states as well.

iv) Energies

The Hamiltonian of Eq. (41) seems to have the wrong sign for the $a^\dagger_{k,0}a_{k,0}$ term. Applying it to the one-particle state $|p,\lambda=0\rangle$, one finds

$$H|p,0\rangle = -p^0|p,0\rangle \ , \tag{44}$$

so the eigenvalue is negative. However, the expectation value of the Hamiltonian for this state is positive due to the negative inner product of the state with itself,

$$\langle p,0|H|p,0\rangle = -p^0\langle p,0|p,0\rangle = +p^0|\langle p,0|p,0\rangle| \ . \tag{45}$$

v) Lorentz Gauge

Things look bad, but we will see that gauge invariance saves things. However, implementing gauge is a little bit tricky. For example, just forcing the Lorenz gauge condition $\partial_\mu A^\mu = 0$ is not consistent with the ETCRs. Explicitly,

$$[\partial_\mu A^\mu(t,\vec{x}),\ A^\nu(t,\vec{x}')] = i\eta^{\nu 0}\delta^{(3)}(\vec{x}-\vec{x}') \neq 0 \ . \tag{46}$$

2.4 Imposing the Gauge Condition

Recall that we started off with the free QED Lagrangian, added the *gauge fixing* term $-(\partial_\mu A^\mu)/2\xi$ to it as in Eq. (27), and then set $\xi = 1$ (rather than treating it as a genuine Lagrange multiplier). With this modified Lagrangian, we derived the conjugate momenta and the classical Hamiltonian, and we applied the usual canonical quantization procedure to it. The resulting quantum theory seems to have many undesirable properties.

The way to fix up the theory to describe physical photons is to find a way to impose the gauge condition $\partial_\mu A^\mu = 0$. Just imposing this condition as an operator equation is too strong, because it conflicts with the ETCRs. Instead, we will only demand that this condition hold in a weakened form when acting on a subset of states in the Hilbert that we will identify with physical configurations of the system. This method is sometimes called the Gupta-Bleuler method after the two people who came up with it.

The key insight of Gupta and Bleuler is that only a subset of the states in the Hilbert should be identified with physical particle excitations. Given a state $|\psi\rangle$, the "physicality" condition is

$$\partial_\mu A^\mu_-|\psi\rangle = 0 \ , \tag{47}$$

where

$$A^\mu_-(x) = \sum_{\lambda=0}^{4}\int \widetilde{dk}\, a_{k,\lambda}\epsilon^\mu(k,\lambda)e^{-ik\cdot x} \ . \tag{48}$$

Note that $A^\mu = A^\mu_- + A^\mu_+$ with $A^\mu_+ = (A^\mu_-)^\dagger$. Applying the derivative to this expansion and using the properties of the polarization 4-vectors, Eq. (47) implies that

$$L_k|\psi\rangle := (a_{k,0} - a_{k,3})|\psi\rangle = 0 \ . \tag{49}$$

Taking the conjugate, we also have $\langle 0|L_k^\dagger = 0$.

The G-B condition is trivially satisfied by the vacuum $|0\rangle$. Going to 1-particle states, $|k,1\rangle$ and $|k,2\rangle$ are both physical excitations, but $|k,0\rangle$ and $|k,3\rangle$ individually are not. The standard terminology is that the $\lambda = 1,2$ are called the *transverse* modes, $\lambda = 3$ is *longitudinal*, and $\lambda = 0$ is *timelike*. More generally, we have for any physical state $|\psi\rangle$

$$\langle\psi|a_{k,3}^\dagger a_{k,3}|\psi\rangle = \langle\psi|a_{k,0}^\dagger a_{k,0}|\psi\rangle \ , \tag{50}$$

as well as

$$\langle\psi|H|\psi\rangle = \sum_{\lambda=1}^{2}\int\widetilde{dk}\, k^0 a_{k,\lambda}^\dagger a_{k,\lambda} \ . \tag{51}$$

Thus, only the transverse modes contribute to the (non-negative) energy of a physical state.

Things are looking up, but there is still the question of how to interpret the longitudinal and timelike polarizations. While neither $|k,0\rangle$ nor $|k,3\rangle$ are physical, there is a particular linear combination of them that is,

$$L_k^\dagger|0\rangle = |k,0\rangle - |k,3\rangle \ . \tag{52}$$

The physicality of this state follows from $[L_k, L_k^\dagger] = 0$. This state also has $HL_k^\dagger|0\rangle = 0$. We interpret this state as being physically equivalent to the vacuum $|0\rangle$ (up to a possible normalization factor). More generally, it can be shown that any physical state can be written in the form [2]

$$|\psi\rangle = G|\psi_T\rangle \ , \tag{53}$$

where $|\psi_T\rangle$ contains only transverse photons (with the same transverse content at $|\psi\rangle$) and the operator G consists exclusively of sums and products of $L_k^\dagger$ operators.[3] Furthermore, one can also show that [2]

$$\langle\psi|A^\mu(x)|\psi\rangle = \langle\psi_T|A^\mu(x)|\psi_T\rangle + \langle\psi_T|\partial^\mu\alpha(x)|\psi_T\rangle \ , \tag{54}$$

for some scalar function α. Thus, we also interpret $|\psi\rangle$ and the corresponding $|\psi_T\rangle$ states as being physically equivalent, and related by a gauge transformation.

2.5 Propagation

To derive Feynman rules, we will need to generalize Wick's theorem to include photon field contractions. As before, the contraction is equal to the time-ordered 2-point function. Applying the mode expansion of $A^\mu(x)$ and the completeness relations for the polarization 4-vectors, the result is

$$\langle 0|T\{A^\mu(x)A^\nu(x')\}|0\rangle = \int\frac{d^4k}{(2\pi)^4}\,\frac{-i\eta^{\mu\nu}}{k^2+i\epsilon}\,e^{-ik\cdot(x-x')} \ . \tag{55}$$

[3]More precisely, $G = 1 + \int\widetilde{dk}_1\, A_1(k_1)L_{k_1}^\dagger + \int\widetilde{dk}_1\int\widetilde{dk}_2\, A_2(k_1,k_2)L_{k_1}^\dagger L_{k_2}^\dagger + \ldots$

We call this the photon propagator.

Two comments about this result are in order. First, all four polarizations contribute to it, which should not be too surprising given that the operator $A^\mu A^\nu$ is not gauge-invariant. Second, this result corresponds to the specific choice of $\xi = 1$ for the gauge fixing parameter. Going through the same procedure for an arbitrary value of ξ, the propagator turns out to be

$$\langle 0|T\{A^\mu(x)A^\nu(x')\}|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2+i\epsilon}\,\left[-\eta^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right] e^{-ik\cdot(x-x')} \ . \tag{56}$$

Our choice of $\xi = 1$ is sometimes called Feynman gauge. Other useful values are $\xi = 0$ (Landau gauge) and $\xi = 3$ (Yennie gauge). We will see shortly that ξ should not appear in any physical observable.

3 Interacting Photons

Let us turn next to QED, with its fermion interactions,

$$\begin{aligned} \mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}i\gamma^\mu D_\mu\Psi - m\bar{\Psi}\Psi & (57)\\ &= (\text{free theory}) - A_\mu\left(eQ\,\bar{\Psi}\gamma^\mu\Psi\right) \ . & (58) \end{aligned}$$

From this, we see that the only coupling in QED connects the vector field to $j^\mu = Q\bar{\Psi}\gamma^\mu\Psi$, the conserved Noether current corresponding to the symmetry of the theory under global rephasing. This has an important implication for scattering matrix elements computed in the theory.

Consider a general QED scattering matrix element with an external photon of momentum k. Given our Feynman rules, the matrix element will have the form (for an incoming photon)

$$\mathcal{M} = \epsilon^\mu(k,\lambda)\,\mathcal{M}_\mu(k) \ , \tag{59}$$

for some quantity $\mathcal{M}_\mu$. Since A_μ always connects with a j^μ in a vertex, we have $\mathcal{M}^\mu(k)$ [4]

$$\mathcal{M}_\mu(k) \sim LSZ\int d^4x\,e^{-ik\cdot x}\langle j_\mu(x)\mathcal{O}\rangle \ , \tag{60}$$

where $\mathcal{O}$ is whatever operator that is needed to make up the rest of the amplitude. This implies that

$$\begin{aligned} k_\mu\mathcal{M}^\mu(k) &= i\int d^4x\,(\partial_\mu e^{-ik\cdot x})\,\langle j^\mu(x)\mathcal{O}\rangle & (61)\\ &= -i\int d^4x\,e^{-ik\cdot x}\langle \partial_\mu j^\mu(x)\mathcal{O}\rangle & (62)\\ &= 0 \ , & (63) \end{aligned}$$

where we have integrated by parts to get the second line. This result, $k_\mu\mathcal{M}^\mu(k)$ is called the *Ward identity.* It is on account of the Ward identity that we can ignore any p_μ terms in the photon polarization sums, so that $\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{\nu *}(p,\lambda) \to -\eta^{\mu\nu}$.

References

[1] D. Tong, "Lectures on Quantum Field Theory".

[2] W. Greiner and J. Reinhardt, Berlin, Germany: Springer (1996) 440 p

[3] M. Srednicki, "Quantum field theory," *Cambridge, UK: Univ. Pr. (2007) 641 p*

[4] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p*