PHYS 526 Notes #7: Fermions and Quantized Spinors

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With the spinor technology in place, we can get to work building a quantum field theory with them. We will find that the only consistent way to do so is to have the spinors represent fermions.

1 Classical Fermions

As a warmup, let's start off by studying the classical equations of motion.

1.1 Classical Weyl

The simplest Lagrangian we can have is

$$\mathscr{L} = \bar{\psi} i \bar{\sigma}^\mu \partial_\mu \psi \ , \tag{1}$$

where $\psi(x)$ transforms in the $(1/2, 0)$ representation of Lorentz. In finding the equations of motion, we should treat ψ and $\bar{\psi}$ as independent variables. This gives

$$\frac{\delta S}{\delta \bar{\psi}_{\dot{\alpha}}(x)} = i(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \psi_\alpha = i(\bar{\sigma} \cdot \partial \, \psi)^{\dot{\alpha}} \tag{2}$$

$$\frac{\delta S}{\delta \psi_\alpha(x)} = -i \partial_\mu \bar{\psi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = -i(\bar{\psi} \, \bar{\sigma} \cdot \overleftarrow{\partial})^\alpha \ . \tag{3}$$

Note that the indices match up nicely. We will suppress them below, but it's a good exercise to check that they all work out.

Let us try to find a plane wave solution of the form

$$\psi_\alpha(x) = u_\alpha(p) e^{-ip\cdot x} \ . \tag{4}$$

Putting this into the equations of motion, we find

$$(\bar{\sigma} \cdot p)\, u(p) = 0 \ . \tag{5}$$

Multiplying by $(\sigma \cdot p)$ and using $(\sigma \cdot p)(\bar{\sigma} \cdot p) = p^2$, we find

$$p^2 = 0 \ . \tag{6}$$

Thus $p^0 = \pm|\vec{p}|$, the dispersion relation of a massless particle. Consider the special case of momentum along the z axis: $p^\mu = k^\mu = (k, 0, 0, k)$. We get

$$0 = (\bar{\sigma} \cdot k) u(k) = k \left[\mathbb{I} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} = k \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} \ , \tag{7}$$

which implies $u_1(k) = 0$. An explicit solution is therefore $\psi(x) = u(k)e^{-ik\cdot x}$ with $k^2 = 0$ and

$$u(k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{8}$$

This coincides with what we would expect for a massless particle with spin anti-parallel to the direction of motion, which is said to be *left-handed*. Repeating these steps for $\bar{\psi}$ yields

$$\bar{\psi}^{\dot{\alpha}} = \bar{u}^{\dot{\alpha}}(k)e^{ik\cdot x} , \tag{9}$$

with

$$\bar{u}(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{10}$$

This corresponds to what we would expect for a *right-handed* particle, with spin parallel to the direction of motion. Note as well that with our choice of normalization,

$$\bar{u}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}(u_\beta)^* . \tag{11}$$

This coincides with the expected relation between ψ and $\bar{\psi}$. It is straightforward to generalize this solution to arbitrary p (with $p^2 = m^2$).

Recall that in `notes-06` we showed that a massless particle with positive energy has a single helicity state while its antiparticle has the opposite helicity. This agrees with the classical solutions we've found for $k^0 > 0$. The spinor ψ is a two-component complex object, and it would naively have four real degrees of freedom. However, applying the equations of motion restricts the components of ψ beyond just the dispersion relation for k^0. This yields two independent solutions that we will identify with a left-handed particle and a right-handed antiparticle when we quantize.[1]

We could do much more with the classical Weyl fermion. However, since we are mainly interested in the electron of QED, which is described by a Dirac fermion consisting of a pair of Weyl fermions, we will move on to that case. For an exhaustive discussion of Weyl fermions, see Ref. [1].

1.2 Classical Dirac

The starting point for the theory of a (classical) Dirac fermion is the Lagrangian

$$\mathscr{L} \;=\; \bar{\Psi}i\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi . \tag{12}$$

To derive the equations of motion, it will be convenient to define an index structure for these four-spinors. We will use

$$\Psi = \Psi_a, \quad \bar{\Psi} = \bar{\Psi}^a, \quad \gamma^\mu = (\gamma^\mu)_a{}^b , \tag{13}$$

[1]The choice of what we label as particle or antiparticle is arbitrary.

where $a = 1, 2, 3, 4$. The equations of motion for Ψ and $\bar{\Psi}$ are

$$\frac{\delta S}{\delta \bar{\Psi}^a(x)} = 0 = (i\gamma^\mu \partial_\mu - m)_a{}^b \Psi_b(x) \ , \tag{14}$$

$$\frac{\delta S}{\delta \Psi_a(x)} = 0 = \bar{\Psi}^a(x)(-i\gamma^\mu \overleftarrow{\partial}_\mu - m)_a{}^b \ . \tag{15}$$

The first equation is called the *Dirac equation* and the second is just its conjugate.

As before, let's look for plane wave solutions to these equations,

$$\Psi_a(x) = a(p)\, u_a(p) e^{-ip\cdot x} \ , \tag{16}$$

where $a(p)$ is a Grassmannian normalization factor and $u_a(p)$ is a four-component column vector of regular (non-Grassmann) numbers, and we will restrict $p^0 > 0$. The Dirac equation then becomes

$$(\gamma^\mu p_\mu - m)u(p) = (\not{p} - m)u(p) = 0 \ . \tag{17}$$

In this expression we have introduced the Feynman slash notation:

$$\not{v} = \gamma^\mu v_\mu = \gamma \cdot v \ , \tag{18}$$

for any four-vector v^μ. Multiplying by $(\not{p} + m)$, we find that

$$(p^2 - m^2)u(k) = 0 \ , \tag{19}$$

which implies $p^0 = \pm\sqrt{\vec{p}^2 + m^2}$, as one would expect for a particle of mass m. Going back to Eq. (17), we can rewrite the result in matrix form as

$$0 = \begin{pmatrix} -m & \sigma \cdot p \\ \bar{\sigma} \cdot p & -m \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix} \ , \tag{20}$$

where we have split u into two two-component pieces. Expanding gives

$$u_R(p) = \frac{\bar{\sigma} \cdot p}{m} u_L(p) \ , \tag{21}$$

which relates the components of $u_L(p)$ and $u_R(p)$ in a non-trivial way. If we plug this into the second equation, we get

$$u_L(p) = \frac{1}{m^2}(\sigma \cdot p)(\bar{\sigma} \cdot p) u_L(p) = \frac{p^2}{m^2} u_L(p) \ , \tag{22}$$

which does not relate the components any further but forces $p^2 = m^2$.

We would like to find a simple basis for these plane wave solutions. It would also be nice if it were somewhat symmetric between u_L and u_R. A convenient way to do this turns out to be [2]

$$u(p, s) = \begin{pmatrix} \sqrt{\sigma \cdot p}\, \xi(s) \\ \sqrt{\bar{\sigma} \cdot p}\, \xi(s) \end{pmatrix} \ , \tag{23}$$

where the square root of the matrix is defined as a formal power series, and $s = 1, 2$ label the two independent basis solutions

$$\xi(1) = \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \qquad \xi(2) = \left(\begin{array}{c} 0 \\ 1 \end{array} \right) . \tag{24}$$

This form looks funny, but it is consistent with Eq. (21); just plug Eq. (23) into Eq. (21) and use $(\sigma \cdot p)(\bar{\sigma} \cdot p) = p^2$:

$$\frac{\bar{\sigma} \cdot p}{m} u_L(p, s) = \frac{1}{m} \sqrt{\bar{\sigma} \cdot p} \sqrt{(\bar{\sigma} \cdot p)(\sigma \cdot p)}\, \xi(s) = \sqrt{\bar{\sigma} \cdot p}\, \xi(s) = u_R(p, s) . \tag{25}$$

Everything works out nicely!

These two solutions have some convenient properties. First, consider them in the rest frame, $p = (m, \vec{0})$, where they reduce to

$$u(p, s) = \sqrt{m} \left(\begin{array}{c} \xi(s) \\ \xi(s) \end{array} \right) , \tag{26}$$

corresponding to spin up or spin down. More generally, in any frame the solutions are orthogonal,

$$\bar{u}(p, s) u(p, r) = u^\dagger(p, s) \gamma^0 u(p, r) = 2m\, \delta^{rs} . \tag{27}$$

Note that the normalization is Lorentz-invariant (just like $\bar{\Psi}\Psi$). The completeness of these solutions implies

$$\sum_{s=1}^{2} u_a(p, s) \bar{u}^b(p, s) = (\gamma \cdot p + m)_a{}^b . \tag{28}$$

We will use this a lot in our calculations.

There are also two other independent solutions for "negative energy", with $\tilde{p}^0 = -\sqrt{\vec{p}^2 + m^2}$. It can be shown that these are equivalent to solutions with $p^0 = +\sqrt{\vec{p}^2 + m^2}$ but with the sign flipped in the exponential:

$$\Psi_a(x) = b^*(p) v_a(p) e^{ip \cdot x} , \tag{29}$$

where $p^0 > 0$ as before. The equations of motion imply

$$-(\gamma^\mu p_\mu + m) v(p) = 0 . \tag{30}$$

A pair of independent solutions are

$$v(p, s) = \left(\begin{array}{c} \sqrt{\sigma \cdot p}\, \eta(s) \\ -\sqrt{\bar{\sigma} \cdot p}\, \eta(s) \end{array} \right) , \tag{31}$$

with $s = 1, 2$ and

$$\eta(1) = \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \qquad \eta(2) = \left(\begin{array}{c} 0 \\ 1 \end{array} \right) . \tag{32}$$

In the rest frame, this reduces to

$$v(p,s) = \sqrt{m} \begin{pmatrix} \eta(s) \\ -\eta(s) \end{pmatrix} . \tag{33}$$

Similarly to before, we also have completeness in the form

$$\sum_{s=1}^{2} v_a(p,s)\bar{v}^b(p,s) = (\gamma \cdot p - m)_a^{\ b} , \tag{34}$$

and orthogonality

$$\bar{v}(p,s)v(p,r) = -2m\, \delta^{sr} . \tag{35}$$

For future calculations it will be useful to have a list of other spinor products:

$$\begin{aligned}
\bar{u}(p,r)u(p,s) &= 2m\, \delta^{rs} && (36)\\
u^\dagger(p,r)u(p,s) &= 2p^0\, \delta^{rs} && (37)\\
\bar{v}(p,s)v(p,r) &= -2m\, \delta^{rs} && (38)\\
v^\dagger(p,s)v(p,r) &= 2p^0\, \delta^{rs} && (39)\\
\bar{v}(p,r)u(p,s) &= 0 && (40)\\
v^\dagger(p,s)u(p,r) &= -2\eta^\dagger(s)(\vec{p}\cdot\vec{\sigma})\xi(r) && (41)\\
v^\dagger(p,r)u(p',s) &= 0, \quad \text{for} \quad p' = (p^0, -\vec{p}) . && (42)
\end{aligned}$$

These are straightforward to prove by simply multiplying the explicit forms of $u(p,s)$ and $v(p,r)$. For example,

$$\begin{aligned}
\bar{v}(p,r)u(p,s) &= \eta^\dagger(r) \left(\sqrt{\bar{\sigma}\cdot p}\sqrt{\sigma\cdot p} - \sqrt{\sigma\cdot p}\sqrt{\bar{\sigma}\cdot p}\right) \xi(s) && (43)\\
&= (\sqrt{p^2} - \sqrt{p^2})\, \eta^\dagger(r)\xi(s) . && (44)
\end{aligned}$$

Pay special attention to the sign flip in Eq. (38) (but not Eq. (39)).

With these basis solutions in hand, we can superpose them to get the most general solution:

$$\begin{aligned}
\Psi(x) &= \sum_{s=1}^{2} \int \widetilde{dp}\ \left[a(p,s)u(p,s)e^{-ip\cdot x} + b^*(p,s)v(p,s)e^{ip\cdot x}\right] && (45)\\
\bar{\Psi}(x) &= \sum_{s=1}^{2} \int \widetilde{dp}\ \left[a^*(p,s)\bar{u}(p,s)e^{ip\cdot x} + b(p,s)\bar{v}(p,s)e^{-ip\cdot x}\right] . && (46)
\end{aligned}$$

In this expression, the exponentials carry the spacetime dependence, the factors of $u(p,s)$ and $v(p,s)$ carry the the four-spinor structure, and the expansion coefficients $a(p,s)$ and $b(p,s)$ make the right side Grassmanian.[2] In all, $\Psi(x)$ is a linear combination of four independent solutions (for each fixed $\vec{p}$) even though it would seem to have eight independent real degrees of freedom. What has happened is that four of these have been eliminated by the algebraic strucutre of the equations of motion. These remaining four coincide with what we would expect for a particle of spin $s = 1/2$ and an antiparticle with the same spin.

[2]Note that u and v are not Grassmann numbers, and thus their components commute.

2 Quantum Fermions

Now we quantize! We will focus on the Dirac case for now.

2.1 Hamiltonian

Before quantizing, let us derive the classical Hamiltonian formulation of the theory defined by the Lagrangian of Eq. (12). The canonical momenta conjugate to $\Psi_a(x)$ and $\Psi^{\dagger\, a}(x)$ are

$$\Pi^a(x) = \frac{\partial \mathscr{L}}{\partial(\partial_0 \Psi_a(x))} = (\bar{\Psi} i\gamma^0)^a = i\Psi^{\dagger\, a} \ , \tag{47}$$

$$\bar{\Pi}_a(x) = \frac{\partial \mathscr{L}}{\partial(\partial_0 \Psi^{\dagger\, a}(x))} = 0 \ . \tag{48}$$

These look a bit funny, but this is just the result of having a kinetic term that is linear in derivatives. What they imply is that we only need to worry about the canonical pair Ψ and $i\Psi^\dagger$ when we quantize later on.

The Hamiltonian of the theory is therefore

$$H = \int d^3\ \mathscr{H} = \int d^3x\ (\Pi^a \partial_0 \Psi_a - \mathscr{L}) \tag{49}$$

$$= \int d^3x\ \left(-\bar{\Psi} i\gamma^k \partial_k \Psi + m\bar{\Psi}\Psi\right) \ , \tag{50}$$

where $k = 1, 2, 3$ is summed over. This looks funny as well, but it is also fine.

2.2 Hamilton, Dirac, and Fermi Walk into a (h)Bar...

To quantize the theory, we elevate $\Psi_a(x)$ and $\Pi^a(x)$ to operators on a Hilbert space and impose equal-time anticommutation relations,

$$\{\Psi_a(t, \vec{x}), \Pi^b(t, \vec{x}')\} = i\delta_a^{\ b}\ \delta^{(3)}(\vec{x} - \vec{x}') \ . \tag{51}$$

We will argue why we should use anticommutators rather than commutators later on. Using our explicit expression for Π^a, Eq. (51) is equivalent to

$$\{\Psi_a(t, \vec{x}), \Psi^{\dagger\, b}(t, \vec{x}')\} = \delta_a^{\ b}\ \delta^{(3)}(\vec{x} - \vec{x}') \ . \tag{52}$$

Let us now *define* the operators $a(p, s)$ and $a^\dagger(p, s)$ (and their conjugates) via [3]

$$a(p, s) = \int d^3x\ e^{ip\cdot x} \bar{u}(p, s)\gamma^0 \Psi(x)\ |_{t=0} \ , \tag{53}$$

$$a^\dagger(p, s) = \int d^3x\ e^{-ip\cdot x} \bar{\Psi}(x)\gamma^0 u(p, s)\ |_{t=0} \ , \tag{54}$$

as well as $b(p,s)$ and $b^\dagger(p,s)$ by [3]

$$
\begin{aligned}
b(p,s) &= \int d^3x \; e^{ip\cdot x}\bar{\Psi}(x)\gamma^0 v(p,s)\,|_{t=0} \;, & (55)\\
b^\dagger(p,s) &= \int d^3x \; e^{-ip\cdot x}\bar{v}(p,s)\gamma^0 \Psi(x)\,|_{t=0} \;. & (56)
\end{aligned}
$$

In all these expressions, we have implicitly set $p^0 = +\sqrt{\vec{p}^2+m^2}$. With some work, one can show that if we were to apply these operations to the right-hand sides of Eqs. (45,46), valid for the classical theory, we would indeed extract the a and b mode functions.

With the definitions of Eqs. (53,54,55,56), the anticommutation relations of Eq. (51) imply very simple relations for the a's and b's :

$$
\begin{aligned}
\{a(k,s), a(p,r)\} &= 0 = \{b(k,s), b(p,r)\} = \{a(k,s), b^{(\dagger)}(p,r)\} & (57)\\
\{a(k,s), a^\dagger(p,r)\} &= (2\pi)^3 2p^0 \delta^{(3)}(\vec{k}-\vec{p})\;\delta^{rs} = \{b(k,s), b^\dagger(p,r)\} & (58)
\end{aligned}
$$

These should remind you of the complex scalar theory you encountered in `hw-01`.

Just like for scalar fields, we will define the vacuum of the theory $|0\rangle$ to be the unique state annihilated by every $a(k,s)$ and every $b(k,s)$ operator,

$$
a(k,s)|0\rangle = 0 = b(k,s)|0\rangle \;. \tag{59}
$$

All other states can be obtained by applying powers of $a^\dagger(k,s)$ and $b^\dagger(k,s)$ to the vacuum. This gives general states of the form

$$
\begin{aligned}
|k_1,s_1;\ldots;k_N,s_N : p_1,r_1;\ldots;p_M,r_M\rangle \;=& & (60)\\
& a^\dagger(k_1,s_1)\ldots a^\dagger(k_N,s_N) b^\dagger(p_1,r_1)\ldots b^\dagger(p_M,r_M)|0\rangle \;.
\end{aligned}
$$

Unlike for scalars, however, the anticommutation relations imply that

$$
[a^\dagger(k,s)]^2 = 0 = [b^\dagger(p,r)]^2 \;. \tag{61}
$$

This means we can only put $n = 0$ or 1 particles in a given state with momentum p and spin s. This coincides precisely with the defining property of fermions that no two of them can occupy the same quantum state. We also see that there are two species of particles, corresponding to $a^\dagger$ and $b^\dagger$. We will interpret them as a particle and an antiparticle.

In terms of the a and b operators, we can invert the relations of Eqs. (53,54,55,56) and solve for Ψ and $\bar{\Psi}$ at $t = 0$:

$$
\begin{aligned}
\Psi(0,\vec{x}) &= \sum_{s=1}^{2}\int \widetilde{dk}\;\left[a(k,s)u(k,s)e^{i\vec{k}\cdot\vec{x}} + b^\dagger(k,s)v(k,s)e^{-i\vec{k}\cdot\vec{x}}\right] & (62)\\
\bar{\Psi}(0,\vec{x}) &= \sum_{r=1}^{2}\int \widetilde{dp}\;\left[a^\dagger(p,r)\bar{u}(p,r)e^{-i\vec{p}\cdot\vec{x}} + b(p,r)\bar{v}(p,r)e^{i\vec{p}\cdot\vec{x}}\right] & (63)
\end{aligned}
$$

With these in hand, we can now write the Hamiltonian in terms of the a and b operators. The calculation is somewhat involved, but the result is

$$H = \sum_{s=1}^{2}\int\widetilde{dk}\;k^0\left[a^\dagger(k,s)a(k,s) - b(k,s)b^\dagger(k,s)\right] \tag{64}$$

$$= \sum_{s=1}^{2}\int\widetilde{dk}\;k^0\left[a^\dagger(k,s)a(k,s) + b^\dagger(k,s)b(k,s)\right] - (\text{constant})\ . \tag{65}$$

As before, we drop the constant by adding a constant to the Lagrangian to cancel it off. This result also shows why we needed anticommutators instead of commutators. The relative sign in Eq. (64) is flipped in going to Eq. (65) by anticommuting the b's. If they had been commutators instead, the minus sign would not have been eliminated. With such a minus sign, states of the form $b^\dagger(k,s)|0\rangle$ would contribute negatively to the energy by an amount $-k^0$. Since this would be physically disastrous – by adding more and more antifermions we could lower the energy arbitrarily – we need anticommutators instead.

To obtain expressions for the fields at arbitrary times, we implement time evolution with the Hamiltonian. It is not hard to check that

$$e^{iHt}a(k,s)e^{-iHt} = e^{-ik^0t}a(k,s), \qquad e^{iHt}b(k,s)e^{-iHt} = e^{ik^0t}b(k,s)\ . \tag{66}$$

This implies that

$$\Psi(t,\vec{x}) = e^{iHt}\Psi(0,\vec{x})e^{-iHt} = \sum_{s=1}^{2}\int\widetilde{dk}\;\left[a(k,s)u(k,s)e^{-ik\cdot x} + b^\dagger(k,s)v(k,s)e^{ik\cdot x}\right] \tag{67}$$

$$\bar{\Psi}(t,\vec{x}) = e^{iHt}\bar{\Psi}(0,\vec{x})e^{-iHt} = \sum_{s=1}^{2}\int\widetilde{dk}\;\left[a^\dagger(k,s)\bar{u}(k,s)e^{ik\cdot x} + b(k,s)\bar{v}(k,s)e^{-ik\cdot x}\right]. \tag{68}$$

It is also clear that the $(N+M)$-particle states we constructed above are eigenstates of the Hamiltonian with total energy $E = (\sum_{i=1}^{N}k^0 + \sum_{j=1}^{M}p_j^0)$.

The identification of $a^\dagger$ as creating a fermion and $b^\dagger$ an antifermion can be illuminated by looking at the $U(1)$ global symmetry of the theory. The Lagrangian of Eq. (12) is symmetric under the transformations

$$\Psi(x) \to e^{-iq\varphi}\Psi(x), \qquad \bar{\Psi}(x) \to e^{iq\varphi}\bar{\Psi}(x)\ , \tag{69}$$

for any constant transformation parameter φ. The conserved current can be shown to be

$$j^\mu = q\,\bar{\Psi}\gamma^\mu\Psi\ , \tag{70}$$

and the corresponding conserved charge is

$$Q = \int d^3x\; j^0 = q\sum_s\int\widetilde{dk}\;\left[a^\dagger(k,s)a(k,s) - b^\dagger(k,s)b(k,s)\right]\ . \tag{71}$$

The total charge Q of a state is therefore just q times the number of fermions minus the number of antifermions. Thus, each fermion contributes q to the total charge Q and each antifermion contributes $-q$. Note that we can also interpret $j^\mu/q = (n_\Psi, \vec{j}_\Psi)$, where n_Ψ is the number density of fermions minus antifermions and $\vec{j}_\Psi$ is the spatial number current.

2.3 Propagating

For fermions, we modify the definition of time ordering to [2]

$$T\{\Psi(x) \overset{(-)}{\Psi}(x')\} = \begin{cases} \Psi(x) \overset{(-)}{\Psi}(x') & ; \quad t > t' \\ -\overset{(-)}{\Psi}(x')\,\Psi(x) & ; \quad t < t' \end{cases} . \tag{72}$$

The additional minus sign reflects the fermionic nature of Ψ and $\bar{\Psi}$, and allows for a smooth limit with $t \to t'$.

With this definition, we can compute all the two-point functions using the mode expansion of Eq. (67). It is immediately obvious that

$$\langle 0|T\{\Psi_a(x)\Psi_b(x')\}|0\rangle = 0 = \langle 0|T\{\bar{\Psi}^a(x)\bar{\Psi}^b(x')\}|0\rangle \ . \tag{73}$$

On the other hand, the mixed two-point function is non-zero. We find

$$\langle 0|T\{\Psi_a(x)\bar{\Psi}^b(x')\}|0\rangle = \int \widetilde{dk} \, \Big[\Theta(t-t')(\not{k}+m)e^{-ik(x-x')} \tag{74}$$

$$-\Theta(t'-t)(\not{k}-m)e^{ik(x-x')}\Big] \ . \tag{75}$$

Here, we have $k^0 = \sqrt{\vec{k}^2 + m^2}$. We can rewrite it as an expression with $k^0 \in (-\infty, \infty)$ using the same contour integral tricks as the scalar:

$$\langle 0|T\{\Psi_a(x)\bar{\Psi}^b(x')\}|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k}+m)_a{}^b}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-x')} \tag{76}$$

$$:= [S_F(x-x')]_a{}^b \ , \tag{77}$$

where $S_F(x-x')$ is called the *Feynman propagator.*

For future reference, let us take the Fourier transform of the two-point function,

$$\int d^4x \, e^{ip\cdot x} \int d^4x' \, e^{ip'\cdot x'} \, \langle \Psi_a(x)\bar{\Psi}^b(x')\rangle = (2\pi)^4\delta^{(4)}(p+p') \frac{i(\not{p}+m)_a{}^b}{p^2 - m^2 + i\epsilon} \tag{78}$$

$$= (2\pi)^4\delta^{(4)}(p+p') \left[\tilde{S}_F(p)\right]_a{}^b \ . \tag{79}$$

Note that

$$(\not{k}-m)(\not{k}+m) = (k^2 - m^2 + i\epsilon) \ . \tag{80}$$

For this reason, $\tilde{S}_F(p)$ is sometimes written as

$$\left[\tilde{S}_F(p)\right]_a{}^b = \left[\frac{i}{\not{k}-m}\right]_a{}^b \ , \tag{81}$$

where this notation is just a shorthand for the fact that the propagator is the inverse $(\not{k}-m)$.

The Feynman propagator also has a nice physical interpretation. For $t > t'$, the only contribution comes from $\langle 0|aa^\dagger|0\rangle$. This coincides with a fermion with charge $+q$ propagating forward in time from x' to x. On the other hand, the only contribution for $t < t'$ comes from $\langle 0|bb^\dagger|0\rangle$, corresponding to an antifermion with charge $-q$ propagating forward in time from x to x'. Both possibilities are described by the amplitude $S_F(x - x')$. Now, for $t > t'$ this amplitude contains a time propagation factor of $\exp(-ip^0|t-t'|)$. However, for $t < t'$ we have a factor of $\exp(+ip^0|t-t'|)$, which seems to have the wrong sign. The insight of Feynman and Stückelberg was that a fermion travelling forward in time from x' to x is indistinguishable from an antifermion travelling backwards in time from x' to x. In both cases a charge $+q$ begins at $\vec{x}'$ and finishes at $\vec{x}$. This accounts for the opposite sign on the exponential factor for $t < t'$.

References

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