

# PHYS 526 Notes #0: Notation and Background

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## 1 Notational Conventions

A large fraction of this course will deal with highly relativistic systems. For this reason, we will use a notation and a set of units that is geared to this situation [1, 2, 3].

### 1.1 Natural Units

We will express quantities in so-called *natural units*, defined by

$$\hbar = c = 1 , \tag{1}$$

where  $\hbar$  is the usual quantum mechanics thing and  $c$  is the speed of light. Since  $\hbar$  has units of energy times time,  $\hbar = 1$  implies that we are measuring time in units of inverse energy. Similarly,  $c = 1$  means we are measuring distance in units of time and therefore units of inverse energy as well. This simplifies dimensional analysis since now all dimensionful quantities can be expressed in units of energy. For example,

$$[E] = [P] = [M] = +1, \quad [L] = [T] = -1, \quad \left[ \frac{d}{dx} \right] = +1, \tag{2}$$

where the square brackets denotes the energy dimension of the quantity (in natural units, of course). The specific unit we will use for energy is the electron Volt (eV), corresponding to the energy acquired by an electron passing through a potential difference of one Volt. We will also use keV =  $10^3$  eV, MeV =  $10^6$  eV, GeV =  $10^9$  eV, and TeV =  $10^{12}$  eV.

To put a result back into regular units, just add powers of  $\hbar$  and  $c$  ( $\sim L/T$ ) until you get what you want. In doing so, it's handy to remember a few things:

$$\hbar c = 1 \simeq 0.197 \text{ GeV} \cdot \text{fm} , \tag{3}$$

$$c = 1 \simeq 3.0 \times 10^{10} \text{ cm} , \tag{4}$$

$$m_p \simeq 0.938 \text{ GeV} \simeq 1.67 \times 10^{-27} \text{ kg} , \tag{5}$$

where  $1 \text{ fm} = 10^{-13} \text{ cm}$ . Despite our usage of natural units, scattering cross sections (which have units of area) will often be expressed in barns (b), with  $1 \text{ b} = 10^{-24} \text{ cm}^2$ . Sometimes it is also convenient to express temperatures in natural units by setting  $k_B = 1$ . This implies  $300 \text{ K} \simeq (1/40) \text{ eV}$  (or about room temperature). Other useful mnemonics are  $m_p \simeq 1 \text{ g}/N_A$  and  $1 \text{ yr} \simeq \pi \times 10^7 \text{ s}$ .

## 1.2 Index Notation – Vectors, Matrices, and More

Index notation will be used a lot in this class, and you'll need to be comfortable with it. As a first example, let's apply it to vectors and matrices.

An  $n$ -component vector  $v$  can be written as an  $n \times 1$  matrix,

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}. \quad (6)$$

Clearly, we can express each of the components of  $v$  as the numbers  $v_i$ ,  $i = 1, 2, \dots, n$ .

The same thing can be done for matrices of any size. For instance, we can write the elements of the  $n \times n$  matrix  $M$  as  $M_{ij}$ :

$$M = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{pmatrix}. \quad (7)$$

Each entry in the matrix can be written as  $M_{ij}$  and is called a *matrix element*.

We can also use index notation to write the products of vectors and matrices. The dot product of a pair of vectors can be thought of as the matrix product of the transpose of the first with the second:

$$u \cdot v = u^t v = (u_1, u_2, \dots, u_n) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \sum_i u_i v_i. \quad (8)$$

From the last equality, written in index notation, it is obvious that  $u \cdot v = v \cdot u = v^t u$ . Note as well that the label we use for the index that is summed over ( $i$  in this case) does not matter:  $\sum_i u_i v_i = \sum_j u_j v_j$ . For this reason, indices that are summed over are often called *dummy indices*.

The product of an  $n \times n$  matrix  $M$  with a column vector  $v$  is itself a column vector ( $Mv$ ). In components, the product is

$$(Mv)_i = \sum_j M_{ij} v_j. \quad (9)$$

Note here that  $j$  is a dummy index (that we can rename), while  $i$  is a *fixed index* that must match up on both sides of the equation. In this case,  $i$  labels the elements of the vector ( $Mv$ ). **Be careful not to use a fixed index to label a dummy index because you**

will get horribly confused and mistaken! For example

$$(Mv)_i = \sum_k M_{ik}v_k \quad (10)$$

$$\neq \sum_i M_{ii}v_i . \quad (11)$$

The product of two matrices  $M$  and  $N$  is a matrix  $(MN)$  with elements

$$(MN)_{ij} = \sum_k M_{ik}N_{kj} = \sum_\ell M_{i\ell}N_{\ell j} . \quad (12)$$

Here, the  $i$  and  $j$  indices are fixed and must match up on both sides of the equation, while the  $k$  index that is summed over is a dummy index.

Index notation is also useful for expressing various matrix operations:

$$(M^t)_{ij} = M_{ji} , \quad (13)$$

$$(M^*)_{ij} = M_{ij}^* , \quad (14)$$

$$(M^\dagger)_{ij} = M_{ji}^* , \quad (15)$$

$$tr(M) = \sum_i M_{ii} . \quad (16)$$

For the special case of three-vectors, we have a cross product operation that takes a pair of vectors and makes another. It can be written in terms of indices using the *antisymmetric tensor*  $\epsilon_{ijk}$  with

$$\epsilon_{123} = +1 = \epsilon_{231} = \epsilon_{312} , \quad (17)$$

$$\epsilon_{132} = -1 = \epsilon_{213} = \epsilon_{321} , \quad (18)$$

and all other entries equal to zero. This name comes from the fact that you get a factor of  $-1$  whenever you exchange a pair of indices in  $\epsilon$ . The cross product is then given by

$$(a \times b)_i = \sum_{j,k} \epsilon_{ijk} a_j b_k . \quad (19)$$

If you are not convinced by this, work out the components explicitly.

### 1.3 Index Notation - Relativistic

We will mostly study systems that are invariant under special relativity, meaning that the underlying equations of the system take the same form after applying Lorentz transformations (boosts and rotations). Discussing such systems is much easier if we use an index notation appropriate to the underlying mathematical structure. Instead of writing  $(t, x, y, z)$  for a specific point in space and time, we will use

$$x^\mu = (t, x, y, z), \quad \mu = 0, 1, 2, 3 . \quad (20)$$

We call this a (position) 4-vector, and it is useful because the components of  $x^\mu$  transform linearly into each other under Lorentz transformations:

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (21)$$

where  $\Lambda^\mu{}_\nu$  is a transformation matrix corresponding to some combination of boosts and rotations. Below, we will discuss the conditions that  $\Lambda^\mu{}_\nu$  must satisfy to count as a Lorentz transformation.

Any four-component object that transforms according to Eq. (21) is called a 4-vector. A second important example is the momentum 4-vector

$$p^\mu = (E, p^x, p^y, p^z), \quad (22)$$

where  $\vec{p}$  is the spatial (3-) momentum of the system and  $E$  is the energy. Recall that for a relativistic particle of mass  $m$ , we have  $E = \sqrt{m^2 + \vec{p}^2}$ .

For any 4-vector  $a^\mu = (a^0, a^1, a^2, a^3)$  with an upper index, we define a corresponding 4-vector with a lower index by

$$a_\mu = \eta_{\mu\nu} a^\nu, \quad (23)$$

where the two-index object  $\eta_{\mu\nu}$  has components

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (24)$$

In writing Eq. (23), we have also used Einstein's summation convention, where we implicitly sum over repeated indices.<sup>1</sup> In full gory detail, Eq.(23) is equal to

$$a_\mu = \eta_{\mu\nu} a^\nu \quad (25)$$

$$= \sum_{\nu=0}^4 \eta_{\mu\nu} a^\nu \quad (26)$$

$$= \begin{cases} a^0 & ; \mu = 0 \\ -a^i & ; \mu = i = 1, 2, 3 \end{cases} \quad (27)$$

It is really important to keep upper and lower indices distinct because it will turn out that they refer to different transformation properties under Lorentz. It is also conventional to use Greek letters for the indices of four-vectors and Roman letters for the components of three-vectors.

In the same way that we lower indices, we can also use the inverse of  $\eta_{\mu\nu}$ , called  $\eta^{\mu\nu}$ , to raise indices. It is not hard to see that viewed as a matrix, the components of both are the same:  $\eta_{\mu\nu} = \eta^{\mu\nu}$ . By definition of the inverse, we have

$$\eta^{\mu\kappa} \eta_{\kappa\nu} = \delta^\mu{}_\nu, \quad \eta_{\mu\kappa} \eta^{\kappa\nu} = \delta_\mu{}^\nu \quad (28)$$

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<sup>1</sup> Apparently Einstein considered this his greatest contribution to physics.

where  $\delta^\mu_\nu = \delta_\nu^\mu$  is the  $4 \times 4$  unit matrix. Raising an index with  $\eta^{\mu\nu}$  is consistent with our previous definition of lowering in that if we lower an index and then raise it again, we get back what we started with:

$$a^\mu = \eta^{\mu\nu} a_\nu = \eta^{\mu\nu} (\eta_{\nu\kappa} a^\kappa) = \delta^\mu_\kappa a^\kappa = a^\mu . \quad (29)$$

These manipulations might seem trivial, but they actually have a lot of content.

A  $4 \times 4$  matrix  $\Lambda^\mu_\nu$  must satisfy certain conditions for it to be a Lorentz transformation. To describe these conditions, it helps to define a 4-vector dot product by

$$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu \quad (30)$$

$$= a_\nu b^\nu = a^\mu b_\mu \quad (31)$$

$$= a^0 b^0 - \vec{a} \cdot \vec{b} . \quad (32)$$

The defining property of a Lorentz transformation is that it leave the dot product of any two vectors invariant. To be precise, if

$$a^\mu \rightarrow a'^\mu = \Lambda^\mu_\nu a^\nu \quad \text{and} \quad b^\mu \rightarrow b'^\mu = \Lambda^\mu_\nu b^\nu , \quad (33)$$

the matrix  $\Lambda^\mu_\nu$  must be such that

$$a' \cdot b' = a \cdot b . \quad (34)$$

This implies the conditions

$$\eta_{\mu\nu} \Lambda^\mu_\lambda \Lambda^\nu_\kappa = \eta_{\lambda\kappa} . \quad (35)$$

Equivalently,  $\eta_{\nu\lambda} \eta^{\mu\kappa} \Lambda^\lambda_\kappa := \Lambda_\nu^\mu = (\Lambda^{-1})^\mu_\nu$ .

Going back to our definition of the momentum 4-vector, taking its dot product with itself gives

$$p^2 := p \cdot p = E^2 - \vec{p}^2 . \quad (36)$$

For a relativistic particle, this is just equal to  $m^2$ , the square of the particle (“rest”) mass which does not change under Lorentz transformations.

Using the chain rule, it is simple to show that a derivative with respect to  $x^\mu$  transforms like a 4-vector with a lower index. As a result, we will use the simplified notation

$$\partial_\mu := \frac{\partial}{\partial x^\mu} \quad (37)$$

$$\partial^2 := \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_t^2 - \vec{\nabla}^2 \quad (38)$$

This implies

$$\partial_\mu x^\nu = \delta_\mu^\nu . \quad (39)$$

Note that we treat  $t$  and  $\vec{x}$  as independent variables. Thinking ahead to quantum mechanics, we will later make the identification  $P^\mu = (P^0, \vec{P}) \sim i(\partial_t, -\vec{\nabla}) = i\partial^\mu$  as operators in a position-space basis.

## 2 Background Physics

Having established notation, we turn next to a quick review of the essential physics background for this course. If you are not familiar with this material, you should come speak to me as soon as possible.

### 2.1 Electromagnetism

Maxwell's equations [4]:

$$\vec{\nabla} \cdot \vec{E} = \rho \quad (40)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (41)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (42)$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j} \quad (43)$$

where  $\rho$  is the charge density and  $\vec{j}$  is the charge current. In writing the equations in this way, we have implicitly also set  $\epsilon_0 = \mu_0 = 1$  (consistent with  $c = 1/\sqrt{\epsilon_0\mu_0}$ ).

The electric and magnetic fields can be written in terms of electric and magnetic potentials:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}. \quad (44)$$

Down the road, we will combine the electric and magnetic potentials into a single four-vector

$$A^\mu = (\phi, \vec{A}). \quad (45)$$

With these conventions, the electric potential due to a stationary point charge  $Q$  is

$$\phi = \frac{Q}{4\pi r}. \quad (46)$$

### 2.2 Classical Mechanics

In the Lagrangian formulation of classical mechanics, a physical system is described by a set of generalized coordinates  $q_i(t)$ . The time evolution of the system is determined by the action [5]:

$$S = \int_a^b dt L(q_i, \dot{q}_i). \quad (47)$$

In many cases, the Lagrangian  $L$  is equal to  $L = T - V$ . The equations of motion that describe the time evolution of the coordinates  $q_i(t)$  are derived from the principle of stationary action,

namely that the physical configuration is the one that satisfies  $\delta S = 0$  for *any* infinitesimal variation  $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$  (with  $\delta q_i = 0$  for  $t = a, b$ ). Applying this condition to the action gives the equations of motion:

$$\partial_t \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 , \quad (48)$$

for each value of  $i$ . This gives one second-order equation for each coordinate  $q_i(t)$ . The solution is unique once initial values for  $q_i(t)$  and  $\dot{q}_i(t)$  are specified.

A second way to formulate classical mechanics is in terms of a Hamiltonian  $H$  that depends on the coordinates  $q_i(t)$  and  $p_i(t)$  [5]. The Hamiltonian formulation can be derived from the Lagrangian, and vice versa. Given a Lagrangian  $L(q_i, \dot{q}_i)$ , the generalized momenta are defined to be

$$p_i = \frac{\partial L}{\partial \dot{q}_i} . \quad (49)$$

Using these equations, it is possible to write each  $\dot{q}_i$  as a function of the  $p_i$  and  $q_i$ . The Hamiltonian  $H(q_i, p_i)$  is then defined to be

$$H = \sum_i \dot{q}_i p_i - L , \quad (50)$$

where all the  $\dot{q}_i$ s on the right-hand side are to be viewed as functions of the  $q_i$  and  $p_i$ . The equations of motion in this formulation can be written in terms of Poisson brackets, defined according to

$$\{f, g\}_{PB} = \sum_j \left( \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right) . \quad (51)$$

With this definition, the equations of motion are

$$\dot{q}_i = \{q_i, H\}_{PB}, \quad \dot{p}_i = \{p_i, H\}_{PB} . \quad (52)$$

Two useful facts to keep in mind are

$$\{q_i, p_j\}_{PB} = \delta_{ij} , \quad (53)$$

and for any function  $f(q, p, t)$

$$\frac{df}{dt} = \{f, H\}_{PB} + \frac{\partial f}{\partial t} . \quad (54)$$

## 2.3 Quantum Mechanics

The rules of quantum mechanics (QM) for relativistic systems are exactly the same as what you have learned in previous classes. However, the procedure we will follow to formulate a consistent quantum theory of relativistic particles will probably be new to you. In most QM courses, you are typically given a system with a Hamiltonian and asked to solve for the corresponding energy eigenvalues and eigenstates. Instead, the major challenge to be addressed in this course will be figuring out how to build the system itself and to deduce what its Hamiltonian should be. So, even though the way we approach things might seem new and unusual, keep in mind that the basic underlying rules are the same as ever.

Recall that a quantum mechanical system consists of a set of states (comprising a Hilbert space) together with the set of operators acting on them [6]. Starting from a classical system defined in terms of  $q_i$ ,  $p_j$ , and  $H$ , we can construct a corresponding quantum system by elevating  $q_i$  and  $p_j$  to operators on the Hilbert space satisfying

$$[q_i, p_j] = i \delta_{ij} . \quad (55)$$

Notice the similarity to the Poisson brackets. The Hamiltonian  $H$  is just a function of the  $q_i$  and  $p_j$ , and is now also a well-defined operator on the Hilbert space of states (up to possible ambiguities in the order of the  $q_i$  and  $p_j$ ).

Quantum states can be viewed as vectors in a (possibly infinite-dimensional) *vector space*, and observables correspond to operators on the space. The vector spaces that arise in quantum mechanics also have an *inner product* defined on them. A key result in linear algebra that we will use a lot is that the eigenstates of a Hermitian operator form a complete basis for the vector space. For example, suppose  $\{|n\rangle\}$  are the eigenstates of the Hamiltonian operator  $H$ , with  $H|n\rangle = E_n|n\rangle$ . Completeness means that any state can be expanded in terms of them:

$$|\psi\rangle = \sum_n a_n |n\rangle , \quad (56)$$

where the coefficients  $a_n$  are just numbers. We can also choose to arrange the energy eigenstates to be orthonormal,  $\langle m|n\rangle = \delta_{mn}$ . In this case, we have

$$a_n = \langle n|\psi\rangle . \quad (57)$$

A convenient way to express completeness and orthonormality (which applies to any set of orthonormal basis states) is the *resolution of the identity*:

$$\mathbb{I} = \sum_n |n\rangle\langle n| , \quad (58)$$

where  $\mathbb{I}$  is the identity operator. Acting the identity, written in this form, on  $|\psi\rangle$  then returns Eq. (56) with the coefficients as in Eq. (57).

Time evolution in quantum mechanics is determined by the Hamiltonian. There are two popular ways to describe this. The first is the Schrödinger picture in which the operators are time-independent while the state of the system evolves according to

$$|\alpha(t)\rangle = e^{-iH(t-t_0)}|\alpha(t_0)\rangle , \quad (59)$$



where  $t_0$  is some reference time. Equivalently, states satisfy the Schrödinger equation,

$$\frac{d}{dt}|\alpha(t)\rangle = -iH|\alpha(t)\rangle . \quad (60)$$

The second standard way to describe time evolution in QM is the Heisenberg picture. Here, the states are time independent while the operators depend on time. Specifically, we have  $|\alpha(t)\rangle = |\alpha(t_0)\rangle$  together with

$$\mathcal{O}(t) = e^{iH(t-t_0)}\mathcal{O}(t_0)e^{-iH(t-t_0)} . \quad (61)$$

This relation is equivalent to the commutation relation

$$\frac{d}{dt}\mathcal{O} = -i[\mathcal{O}, H] , \quad (62)$$

which should remind you of the classical Poisson bracket result.

This discussion should be very familiar from your previous courses on QM, so let's try something a bit different. Specifically, let's build a Hilbert space for a system and figure out what kinds of operators act on it.

### ***e.g.* 1. Particles in Zero Dimensions**

Consider a system consisting of a box that can contain any number of identical (bosonic), non-interacting particles at rest. Let us assume further that each particle contributes an energy  $M$  to the total. A complete set of states is therefore  $\{|n\rangle : n = 0, 1, 2, \dots\}$ , where  $n$  labels the number of particles in the box. The total energy of state  $|n\rangle$  is  $nM$ , so we have

$$H|n\rangle = nM|n\rangle .$$

At this point, we are basically done. We have a complete Hilbert space for the system and we can figure out the time evolution of any observable by applying the Schrödinger equation.

Even so, let us fiddle a bit more and look into what sorts of operators can act on the space. Just about the only thing we can do to the system is add or remove particles, so it's worth thinking about operators that would accomplish this. Let us *define* an operator that removes particles one at a time by

$$a|n\rangle = \sqrt{n}|n-1\rangle .$$

There is nothing stopping us from making such a definition; this is a perfectly well-defined operator on the Hilbert space. Taking the Hermitian conjugate of this and contracting the result with all possible kets, we also find

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle .$$

From this, we can also derive

$$[a, a^\dagger] = 1 ,$$

as well as

$$H = M a^\dagger a .$$

In fact, it can be shown that any operator whatsoever on the Hilbert space can be built up from sums and products of  $a$  and  $a^\dagger$

All this should be familiar from the case of the harmonic oscillator. Now, however, the system does not correspond to a single point particle in a quadratic potential. Even so, we can rewrite the Hamiltonian in terms of  $X$  and  $P$  operators. For this, let us take the two linear combinations of  $a$  and  $a^\dagger$  that are Hermitian:

$$X = (a + a^\dagger), \quad P = -i(a - a^\dagger) .$$

These relations are invertible, so any operator on the Hilbert space can be constructed from  $X$  and  $P$  as well. In particular, we have

$$H = M(X^2 + P^2)/2 - M/2 . \tag{63}$$

Thus, our system of identical stationary bosons can be described by continuous generalized coordinate operators  $X$  and  $P$  that don't have anything to do with space or velocity.

### 3 Useful Math

The mathematics to be used in this course will consist mostly of vector calculus, together with a bit of group theory. We will get to the group theory later on, so let us just mention a few useful results in vector calculus [7].

We will write  $n$ -dimensional integrals over spacetime (one time and  $(n - 1)$  spatial dimensions) as

$$\int d^n x = \int dt \int dx^1 \int dx^2 \dots \int dx^{n-1} \tag{64}$$

The divergence theorem generalizes to this case: for any  $n$ -dimensional volume  $V$  with boundary  $\partial V$ , we have

$$\int_V d^n x \partial_\mu f^\mu = \int_{\partial V} dA n_\mu f^\mu , \tag{65}$$

where  $n_\mu$  is the outwardly pointing normal unit vector to the bounding surface  $\partial V$ . In most cases of interest to us, the surface integral over  $\partial V$  will be zero for one reason or another.

Multi-dimensional delta functions will appear frequently:

$$\int d^n x \delta^{(n)}(x - x') f(x) = f(x') . \tag{66}$$

Equivalently  $\delta^{(n)}(x - x') = \prod_{\mu=0}^{n-1} \delta(x^\mu - x'^\mu)$ . We also have the Fourier transform relation

$$\int d^n x e^{ik \cdot x} = (2\pi)^n \delta^{(n)}(k), \quad \int d^n k e^{ik \cdot x} = (2\pi)^n \delta^{(n)}(x). \quad (67)$$

Finally, recall that

$$\int dx \delta(f(x)) g(x) = \sum_a g(x_a) / |df/dx|_{x_a}, \quad (68)$$

where  $x_a$  refers to any value of  $x$  such that  $f(x_a) = 0$ .

Sometimes it will be useful to use contour integration in the complex plane. The key result is Cauchy's theorem, which is that for any integral around a closed counterclockwise contour  $C$  in the complex plane,

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)}, \quad (69)$$

provided  $f(z)$  has no poles in the region bounded by the contour. If the contour runs clockwise, the formula picks up an overall minus sign. Differentiating both sides with respect to  $z_0$  gives

$$\frac{d^n}{dz^n} f(z_0) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)^{n+1}}. \quad (70)$$

This is really useful because it also applies to any deformation of the contour  $C$  that does not intersect the pole. Sometimes the stuff in the integrand (with the pole factored off) is called the *residue* of the pole. Cauchy's theorem is therefore often stated as

$$\oint_C dz f(z) = 2\pi i \times (\text{sum of residues}). \quad (71)$$

**e.g. 2.**  $I = \int_0^\infty dx 1/(1 + x^2)$

We can do this integral easily using Cauchy's theorem. See Fig. 1 for an illustration of the contours  $C_1$  and  $C_2$ .

$$\begin{aligned} I &= \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{1}{1 + x^2} \\ &= \frac{1}{2} \lim_{R \rightarrow \infty} \int_{-R}^{+R} dx \frac{1}{1 + x^2} = \lim_{R \rightarrow \infty} \int_{C_1} dz \frac{1}{1 + z^2} \\ &= \frac{1}{2} \lim_{R \rightarrow \infty} \left[ \oint_{C=C_1+C_2} dz \frac{1}{(z+i)(z-i)} - \int_{C_2} dz \frac{1}{1 + z^2} \right] \\ &= \frac{1}{2} \times 2\pi i \times \frac{1}{2i} - 0 \\ &= \frac{\pi}{2}. \end{aligned}$$

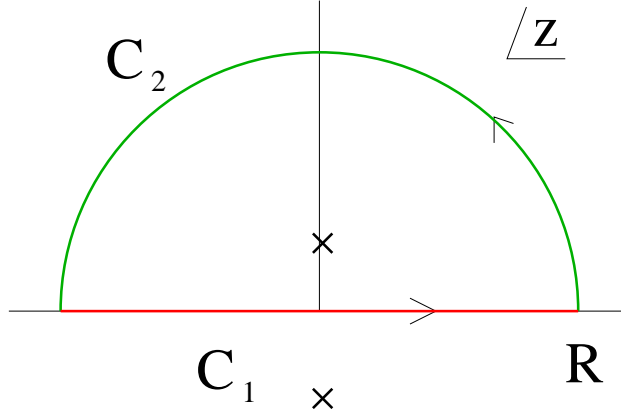


Figure 1: Contours for *e.g.* 2.

In the last line, we have used the fact that the contour integration along  $C_2$  can be parametrized by  $z = Re^{i\theta}$  and goes like

$$\lim_{R \rightarrow \infty} \int_{C_2} dz \frac{1}{1+z^2} = \lim_{R \rightarrow \infty} \int d\theta iRe^{i\theta} \frac{1}{1+R^2e^{2i\theta}} \rightarrow 0 .$$

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