PHYS 526 Homework #6

Due: Oct. 22, 2012

0. Read Ch.3 of Peskin & Schroeder, Chs.2, 33, 34 of Srednicki, and my `notes-05`.

1. $SU(2)$ representations:
   (Except for part g), you should only use the Lie algebra of $SU(2)$ for this question.)

   a) Show that $C_2 = t^a t^a$ (summed on a) commutes with all three generators.

   b) Define $t^{\pm} = t^1 \pm i\, t^2$. Find their commutation relations with each other and t^3.

   c) Suppose we have a simultaneous eigenvector $|C, m\rangle$ of C_2 and t^3 with eigenvalues C and m. Show that $t^{\pm}|C, m\rangle$ is also an eigenvector of both C_2 and t^3 and find its eigenvalues.

   d) For a finite representation, we need $t^-|b\rangle = 0$ for some joint eigenstate $|b\rangle$. Let us call the t^3 eigenvalue $-j$ for some $j \in \mathbb{R}$. Find the C_2 eigenvalue of this state in terms of j. *Hint: write t^+t^- in terms of C and t^3 and apply it to $|b\rangle$.*

   e) Apply t^+ n times to $|b\rangle$. What are the C_2 and t^3 eigenvalues?

   f) For a finite representation, we must eventually reach a state with $t^+|t\rangle = 0$. What are the C_2 and t^3 eigenvalues in terms of j? What does this imply for the allowed values of j?

   g) Use this technology to construct explicit matrix representations for the generators of the $SU(2)$ Lie algebra for $j = 0$, $j = 1/2$ and $j = 1$. In each case, take $(0, \ldots, 1)^t$ as the bottom state.
   Hint: be careful to normalize the states correctly!

2. Construct a complex scalar field theory that is invariant under global $SU(2)$ transformations with the fields $\Phi(x)$ transforming under the $j = 1$ representation of the group. Explain how the field transforms, and write out an invariant Lagrangian with non-trivial interactions.

3. Lorentz:

   a) Show that $\delta\omega_{\mu\nu}$ must be antisymmetric.

   b) Verify that $(J_4^{\mu\nu})_{\alpha\beta}$ corresponds to the vector representation as claimed.

   c) Show that ∂_μ transforms as a $(0, 1)$ tensor.

   d) Given the commutators of J^i and K^i, work out the commutators of A^i and B^i defined in the notes.

   e) Work out the infinitesimal forms of the transformation matrices $\Lambda^\mu{}_\nu$ when only ω_{12} is non-zero and when only ω_{01} is non-zero. What do these correspond to?

4. Derive $[P^\mu, J^{\rho\sigma}]$ by expanding Eq. (66) of `notes-05` to linear order for $\Lambda = 1 + \omega$.
   Hint: $\omega_{\alpha\beta} = \frac{1}{2}(\omega_{\alpha\beta} - \omega_{\beta\alpha})$.

5. Work out the explicit representation matrices for J^i and K^i acting on a field transforming in the $(0, 1/2)$ rep of Lorentz. How does this field transform under finite Lorentz transformations?