PHYS 526 Homework #3

Due: Oct. 1, 2013

0. Read Chs. 4.1-4.3 of Peskin & Schroeder, Chs. 4-5 of Srednicki, and Ch. 3.1-3.4 of Tong. You can find the last here: `www.damtp.cam.ac.uk/user/tong/qft/qft.pdf` .

1. D_F as a Green's function.

   a) By differentiating the expression in Eq. (41) of `notes-02`, show that

   $$(\partial^2_{(x)} + m^2)\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = -i\delta^{(4)}(x-y) \ ,$$

   where $\partial^2_{(x)}$ means that you should differentiate with respect to the components of $x = (x^0, \vec{x})$ rather than y.

   b) Show that $\frac{d}{dx}\Theta(x-x') = \delta(x-x')$, in the sense that $\int_{-\infty}^{\infty} dx\, f(x) \frac{d}{dx}\Theta(x-x') = f(x')$ for any reasonable function $f(x)$. Show also that $\frac{d}{dx}\Theta(x'-x) = -\delta(x-x')$.

   c) Use this result to prove the Green's function relation above in a second way: apply $(\partial^2_{(x_1)} + m^2)$ to Eq. (37) of `notes-02`, and make use of $(\partial^2 + m^2)\phi(x) = 0$ (via Eqs.(28, 29) of `notes-02`). (*Hint:* $f \cdot \partial_{(1)}\delta(t_1 - t_2) = -\delta(t_1-t_2)\cdot\partial_{(1)} f$, $\Pi = \partial_t \phi$.)

2. Consider the quantum theory of a complex scalar field with Lagrangian

   $$\mathscr{L} = |\partial\Phi|^2 - m^2|\Phi|^2 - \Lambda \ .$$

   The fields $\Phi(x)$ and $\Phi^*(x)$ should be thought of as independent degrees of freedom. This is a free theory, and we can expand the fields exactly in terms of two independent mode operators $a(\vec{k})$ and $b(\vec{k})$:

   $$\begin{aligned} \Phi(x) &= \int \widetilde{dk}\left[a(\vec{k})e^{-ik\cdot x} + b^\dagger(\vec{k})e^{ik\cdot x}\right] \\ \Phi^\dagger(x) &= \int \widetilde{dk}\left[b(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right] \ , \end{aligned}$$

   where $k^0 = \sqrt{\vec{k}^2 + m^2}$ and the mode operators satisfy

   $$\begin{aligned} [a(\vec{k}), a(\vec{p})] &= 0 = [b(\vec{k}), b(\vec{p})] = [a(\vec{k}), b(\vec{p})] = [a(\vec{k}), b^\dagger(\vec{p})] \\ [a(\vec{k}), a^\dagger(\vec{p})] &= (2\pi)^3 2p^0 \delta^{(3)}(\vec{p}-\vec{k}) = [b(\vec{k}), b^\dagger(\vec{p})] \ . \end{aligned}$$

   a) Assuming a unique vacuum $|0\rangle$ annihilated by all the a's and b's, build up the Hilbert space in terms of energy-momentum eigenstates. Interpret physically.

   b) Use the commutation relations to evaluate the 2-point functions for $\Phi(x_1)\Phi(x_2)$, $\Phi^\dagger(x_1)\Phi^\dagger(x_2)$, $\Phi^\dagger(x_1)\Phi(x_2)$, and $\Phi(x_1)\Phi^\dagger(x_2)$. You should express your results in terms of the Feynman propagator $D_F(x_1 - x_2)$ we found for the real scalar theory.

   c) Write $\Pi = \partial_0\Phi^\dagger$ and $\Pi^\dagger = \partial_0\Phi$ in terms of the mode operators, and compute $[\Phi(t,\vec{x}), \Pi(t,\vec{y})]$ and $[\Phi(t,\vec{x}), \Pi^\dagger(t,\vec{y})]$ by using the mode-operator commutators.

d) Recall from `hw-01` that this theory has a conserved charge corresponding to a symmetry under phase rotations. Show that the corresponding charge operator is

$$Q = \int \widetilde{dk}\, \left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right] \ .$$

The physical interpretation is that $a^\dagger$ creates positively charged particles and $b^\dagger$ creates negatively charged ones. What is the net charge of the states you found in part a)?

3. Consider the theory of a real scalar defined by the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \Lambda - \frac{g}{3!}\phi^3 \ .$$

a) In the free theory ($g = 0$), show that $\langle \vec{p}_2, \vec{p}_3|e^{-iHt}|\vec{p}_1\rangle = 0$.

b) In the full theory ($g \neq 0$), show that this matrix element can be non-zero (where $|\vec{p}_1\rangle = a^\dagger(\vec{p}_1)|0\rangle$ and $\langle \vec{p}_2, \vec{p}_3| = \langle 0|a(\vec{p}_2)a(\vec{p}_3)$ are defined in terms of the H_0 basis states and ladder operators constructed in `notes-3` at $t = 0$).
Hint: look at small t and expand to linear order, and then use the mode expansion of $\phi(0, \vec{x})$. It is enough to show that you can balance the total number of a operators with the total number of $a^\dagger$ operators.

4. Time Ordering.

a) What is $[T\{\phi(x_1)\phi(x_2)\}]^\dagger$ when ϕ is a real scalar field?.
Hint: write out the definition of time ordering in terms of step functions and take the Hermitian conjugate of that. Also, $(AB)^\dagger = B^\dagger A^\dagger$.

b) Find and define explictly an operator operation T' analogous to time ordering such that (for the real field ϕ)

$$T'\{\phi(x_1)\phi(x_2)\} = [T\{\phi(x_1)\phi(x_2)\}]^\dagger \ ,$$

Show that your definition also applies to products of ϕ and Π fields.

c) Show that $[T\{\phi(x_1)\phi(x_2)\}]^\dagger + T\{\phi(x_1)\phi(x_2)\} = \phi(x_1)\phi(x_2) + \phi(x_2)\phi(x_1)$.

d) We defined $U(t) = e^{iH_0t}e^{-iHt}$, where H_0 is the free Hamiltonian and H is the full version, and showed that its time evolution is given by $i\partial_t U(t) = \Delta H_I(t)U(t)$. Show by explicit differentiation that

$$U(t) = \widetilde{T}\left\{\exp\left[-i\int_0^t dt'\Delta H_I(t')\right]\right\} \ ,$$

is a solution to this equation with the correct boundary condition, where $\widetilde{T}$ denotes time ordering for $t > 0$ and reverse time ordering for $t < 0$.

e) What is the equation for the time dependence of $U^\dagger(t)$? Propose a solution valid for $t > 0$ analogous to the one for $U(t)$ and show that it works by differentiating, but possibly with a modified time-ordering prescription.