PHYS 526 Homework #2

Due: Sept. 24, 2013

1. Define the operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ at the fixed reference time t_0 as in Eqs. (15,16) of `notes-02`. Using the canonical commutation relations of $\phi(t_0, \vec{x})$ and $\Pi(t_0, \vec{x})$, show that they imply that $a(\vec{k})$ and $a^\dagger(\vec{k})$ satisfy the commution relations of Eqs. (17,18).

2. Energy in the quantum theory.

   a) Derive the classical expression for the conserved charge corresponding to time translations in our simple free scalar theory using Noether's theorem. Show that it matches the Hamiltonian H for this theory.

   b) Invert Eqs. (15,16) of `notes-02` to write $\phi(t_0, \vec{x})$ and $\Pi(t_0, \vec{x})$ in terms of a and $a^\dagger$ operators.

   c) The H operator in the quantum theory is the same as in the classical theory written in terms of ϕ and Π, but with ϕ and Π elevated to operators. Write $H(t) = H(t_0)$ in the quantum theory in terms of a and $a^\dagger$ operators. Your final result should have only a single $\widetilde{dk}$ integration.

   d) Derive the commutation relations of $a(\vec{k})$ and $a^\dagger(\vec{k})$ with $H(t_0)$.

3. Time evolution.

   a) The a and $a^\dagger$ ladder operators we defined previously were time-independent. We can also define time-dependent versions of them according to

   $$a(t, \vec{k}) = e^{iHt} a(\vec{k}) e^{-iHt} \ ,$$

   and similarly for $a^\dagger$. This implies $\partial_t a(t, \vec{k}) = i[H, a(t, \vec{k})]$. Show that a solution to this operator equation (with the correct boundary condition) is

   $$a(t, \vec{k}) = e^{-ik^0 t} a(\vec{k}) \ .$$

   Derive the corresponding result for $a^\dagger(t, \vec{k})$ as well.

   b) Use this result to extend the expressions for $\phi(t_0, \vec{x})$ and $\Pi(t_0, \vec{x})$ derived above in terms of a and $a^\dagger$ to all times. For notational simplicity, set $t_0 = 0$.
   Hint: recall that $\mathcal{O}(t) = e^{iH(t-t_0)} \mathcal{O}(t_0) e^{-iH(t-t_0)}$ for any local operator $\mathcal{O}(t)$.

4. Spatial Translations.

   a) In the classical free scalar theory, derive the Noether currents $j^{\mu i}$ and the conserved charges P^i corresponding to invariance under spatial translations and express them in terms of $\phi(x)$ and $\Pi(x)$.

   b) The same expressions apply in the quantum theory but with ϕ and Π elevated to operators. There is an ambiguity in how to order the ϕ and Π factors, but for now let us choose to keep all the Π's to the left of all the ϕ's. With this choice,

rewrite the charges P^i in terms of $a(\vec{k})$ and $a^\dagger(\vec{k})$ modes and simplify until you have a single $\widetilde{dk}$ integration. Your result will be time-independent if you've done it right.
Hint: a lot of stuff vanishes by symmetry; $\int \widetilde{dk}\, k^i\, g(\vec{k}) = 0$ *for any function* $g(\vec{k})$ *such that* $g(-\vec{k}) = g(\vec{k})$*.*

c) Apply P^i to $[a^\dagger(\vec{k}_1)]^{n_1}[a^\dagger(\vec{k}_2)]^{n_2}\ldots[a^\dagger(\vec{k}_N)]^{n_N}|0\rangle$ and show that this state is an eigenvector with eigenvalue $\sum_{i=1}^N n_i\vec{k}_i$.

d) Show that $[P^i, \phi(t,\vec{x})] = i\partial_i\phi(t,\vec{x})$ and $[P^i, \Pi(t,\vec{x})] = i\partial_i\Pi(t,\vec{x})$. By composing infinitesimal translations, this is equivalent to

$$\phi(t,\vec{x}+\vec{a}) = e^{-i\vec{P}\cdot\vec{a}}\phi(t,\vec{x})e^{i\vec{P}\cdot\vec{a}}, \quad \Pi(\vec{x}+\vec{a}) = e^{-i\vec{P}\cdot\vec{a}}\Pi(t,\vec{x})e^{i\vec{P}\cdot\vec{a}} \ .$$

Thus, $\vec{P}$ generates spatial translations in the quantum theory as well.

e) Combine this result with what we know about time evolution to show that:

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x)$$

as well as

$$\phi(x+a) = e^{iP\cdot a}\phi(x)e^{-iP\cdot a} \ .$$

Unsurprisingly, the operator P^μ is called the generator of spacetime translations.

5. Starting from the expansion of $\phi(x)$ in terms of the ladder operators, use the contour integration result you found in `hw-00` (or its generalization) to show that

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = D_F(x_1-x_2) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x_1-x_2)} \ .$$

where ϵ is to be set to zero after doing the dp^0 contour integration. You should treat the $t_1 > t_2$ and $t_1 < t_2$ cases separately.
Hint: for the countour integrals, think carefully about how to close the contour in each of the two cases.