PHYS 526 Homework #0

Due: Sept. 10, 2013

0. Read Ch.1,2 of Peskin and Schroeder.

1. Natural Units

   a) What is one second in GeV units?

   b) What is one meter in GeV units?

   c) The LHC is trying to create new particles with masses M on the order of a TeV. On dimensional grounds, we expect the production cross section for such particles to go like $\sigma \sim 1/M^2$. What does this correspond to in femptobarns ($1\,\text{fb} = 10^{-15}\text{b}$).

   d) The mass scale that corresponds to Newton's constant $G_N = 6.67 \times 10^{-11}\,\text{m}^3/\text{kg}\,\text{s}^2$ is called the Planck mass. What is its value in GeV units?

   e) The age of the Universe is about 13.7 billion years. Express this in GeV units and compare it to the Planck mass you found above.

2. Matrices and Indices

   a) Show that for any two $n \times n$ matrices, $(MN)^t = N^t M^t$.

   b) Show that δ_{ij} $(i, j = 1, \ldots, n)$ is the $n \times n$ identity matrix.

   c) Prove the cyclicity of the trace: $tr(AB) = tr(BA)$.

   d) Show that ϵ_{ijk} is cyclic:

   $$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}\ . \tag{1}$$

   Use this to show that the triple product ($n = 3$) is also cyclic in the sense $a\cdot(b\times c) = c \cdot (a \times b) = b \cdot (c \times a)$.

   e) For $M = \sigma^1$ (the Pauli matrix) and $v^t = (1, 1)$, evaluate $\sum_j M_{ij} v_j$ and $\sum_i M_{ii} v_i$.

3. Relativistic Indices

   a) Show that $\eta_{\mu\nu}\, \Lambda^{\mu}{}_{\lambda}\, \Lambda^{\nu}{}_{\kappa} = \eta_{\lambda\kappa}$ implies that Λ leaves invariant the dot product of any pair of 4-vectors.

   b) Prove $\eta_{\nu\lambda}\, \eta^{\mu\kappa}\, \Lambda^{\lambda}{}_{\kappa} := \Lambda_{\nu}{}^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}$.

   c) Objects with more than one Lorentz index are called tensors. Like vectors, we raise and lower their indices with η (*e.g.* $T^{\mu\nu} = \eta^{\mu\lambda} T_{\lambda}{}^{\nu} = \eta^{\mu\lambda}\eta^{\nu\kappa} T_{\lambda\kappa}$). Under Lorentz transformations, each index gets a power of Λ (*e.g.* $T^{\mu\nu} \to \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}_{\kappa} T^{\lambda\kappa}$).

      i) Show that if we treat it as a tensor, $\eta^{\mu\nu} \to \eta^{\mu\nu}$ under Lorentz transformations.

      ii) A pair of tensor indices are said to be antisymmetric if $A^{\mu\nu} = -A^{\nu\mu}$. Show that if A is antisymmetric, $A^{\mu\nu} v_{\mu} v_{\nu} = 0$.

      iii) Show that $T^{\mu\nu} u_{\mu} v_{\nu}$ is Lorentz invariant for any tensor T and vectors u, v.

4. Classical Mechanics

   a) For a system with coordinates q_i and p_j and Hamiltonian H, use the equations of motion and the defintion of the Poisson bracket to prove that for any $f(q,p,t)$, $df/dt = \{f, H\} + \partial f/\partial t$.

   b) Work out the equations of motion for the classical harmonic oscillator using both the Lagrangian and Hamiltonian formulations. In each case, what are the initial conditions that you need to specify to get a unique solution?

5. Evaluate the integral

$$I_2 = \int d^3p_a \int d^3p_b \, \frac{1}{E_a E_b} \, \delta^{(4)}(P - p_a - p_b) \ ,$$

   where $P = (M, \vec{0})$, $p_a^0 = E_a = \sqrt{m^2 + \vec{p}_a^{\,2}}$, and $p_b^0 = E_b = \sqrt{m^2 + \vec{p}_b^{\,2}}$.

6. Contour Integration #1 - evaluate this integral using contour integration for A and B real and positive.

$$I = \int_{-\infty}^{\infty} dx \, \frac{x}{(x^2 + A^2)(x^2 + B^2)} \ . \tag{2}$$

7. Contour Integration #2 - evaluate this integral using contour integration for E and T real and positive.

$$I = \int_{-\infty}^{\infty} dx \, \frac{e^{-ixT}}{x^2 - E^2 + i\epsilon} \ . \tag{3}$$

   Here, treat $0 < \epsilon \ll E^2$, and set $\epsilon \to 0$ at the end.

   Hint #0: Where are the poles?
   Hint #1: You can close the integration contour in the upper half plane or in the lower half plane. One choice will blow up, the other will go nicely to zero.
   Hint #2: $e^{i\theta} = \cos\theta + i\sin\theta$.