PHYS 526 Final Project

Due: Dec. 12, 2013, 11am

Instructions:

- Due on or before **Thursday, December 12, 2013, at 11am**.
- You can consult any books/notes/websites you like (except for class material from last year), but your final answers should represent your own understanding.
- You are strongly encouraged to ask me anything you like about the project or the course. However, please do not discuss any part of the final project with anyone else.
- Show how you got your result. Points will be assigned for showing how you solved the problem rather than for the final answer itself.
- I will be in Hennings 319 from 9:30-11:00am on Dec. 3, 5, 10, 12 and from 10am-12pm on Nov. 29 and Dec. 6. If you would like to speak at another time, email me and we can arrange a meeting in the department.
- There are five questions and 50 points in total.
- All the standard UBC exam rules apply.

Questions

1. Heavy fermion decay (**10 = 1+1+4+4 points**)
   Consider a theory with two Dirac fermions ψ and χ and a real massive vector V^μ with coupling

   $$-\mathscr{L} \supset g\,V_\mu \left(\bar{\psi}\gamma^\mu P_L \chi + \bar{\chi}\gamma^\mu P_L \psi\right)$$

   Assume $m_\psi \gg m_V$ and $m_\chi = 0$ so that the decay $\psi \to \chi V$ is kinematically allowed.

   a) Show that the Lagrangian term given above is real (for real g).
   b) Estimate the rate for this decay using dimensional analysis.
   c) Compute the summed and squared matrix element for the total decay rate.
   d) Use this to find the total decay rate and compare to your result from part b).

2. Dark Matter Direct Detection (**10 = 2+4+4 points**)
   One way to search for dark matter (DM) is to look for the scattering of DM particles off nuclei in a detector. Suppose the DM particle is a Dirac fermion χ and the nucleus can be treated as a complex scalar N with the interactions

   $$-\mathscr{L} \supset y\,\phi\,\bar{\chi}\chi + A\phi|N|^2 \ ,$$

   where ϕ is a real scalar. The signal rate for dark matter scattering is proportional to the $\chi N \to \chi N$ elastic scattering cross section. Typically, it is an excellent approximation to assume that the target nucleus N is at rest and the incoming DM particle χ is non-relativistic, with a typical velocity of $v \sim 10^{-3}$. We usually also expect that the masses of the DM, the ϕ scalar, and the nucleus are all comparable.

a) Work out the kinematics of this scattering assuming that N begins at rest and χ has initial speed v along the z axis. Specifically, compute the 3-momentum $\vec{q}$ and the energy of the outgoing nucleus in terms of the masses, the initial velocity, and the scattering angle θ of the nucleus relative to the incoming DM direction. Since the incoming χ is non-relativistic, you may work in the non-relativistic limit, corresponding to the approximation $p \simeq mv$ and $E \simeq m + p^2/2m$.

b) Calculate the summed and squared matrix element for this process in terms of $\{p_1, p_2, p_3, p_4\}$, where $N(p_1) + \chi(p_2) \to N(p_3) + \chi(p_4)$. Next, expand your result in powers of v and keep only the leading non-trivial term. Is this a good approximation (to within 1%) of the full matrix element for $m_\chi \sim m_N \sim m_\phi$?

c) Compute the total unpolarized DM-nucleus scattering cross section. You should be able to evaluate all the integrals by using the non-relativistic expansion.

Hint #1: use the spatial part of the delta function to get rid of the d^3p_4 *integrals and rewrite the remaining integrals and integrand in terms of* q *and* $\cos\theta$.

Hint #2: apply your result from a) to remove one of the remaining integrals using the leftover delta function, and remember that $|\cos\theta| \leq 1$.

3. Pion Decay (**10 = 3+3+3+1 points**)
A charged pion π^- is a complex scalar that can decay to a lepton ℓ and an anti-neutrino $\bar{\nu}$. The corresponding interaction can be modelled by

$$-\Delta\mathscr{L} = \frac{1}{f}(i\partial_\mu \pi^-)\,(\bar{\ell}\gamma^\mu P_L \nu) + h.c.$$

a) Find the position-space three-point function $\langle (\pi^-)^*(x_1)\,\ell_a(x_2)\,\bar{\nu}^b(x_3)\rangle$.

b) Take the Fourier transform and use this to deduce the momentum-space Feynman rule for the vertex when the pion momentum is incoming.

c) Compute the total unpolarized decay rate for $\pi^- \to \ell\bar{\nu}$. You can treat the neutrino ν as being massless, but you should keep the dependence on the masses of the pion and the lepton.

d) What does your result imply for the relative probability for a charged pion to decay to an electron rather than a muon?

4. Scalars and Vectors (**10 = 1+1+1+2+2+1+2 points**)
Consider a theory of a complex scalar Φ and a real vector A^μ with Lagrangian density

$$\mathscr{L} = \eta^{\mu\nu}(D_\mu\Phi)^*(D_\nu\Phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\Phi,\Phi^*)\ ,$$

where

$$V(\Phi,\Phi^*) = m^2|\Phi|^2 + \frac{\lambda}{4}|\Phi|^4 \qquad \text{and} \qquad (D_\mu\Phi) = (\partial_\mu + iQeA_\mu)\Phi\ .$$

a) Show that this theory is invariant under the gauge transformation

$$\Phi(x) \to e^{-iQ\,\alpha(x)}\Phi(x), \qquad \Phi^*(x) \to e^{iQ\alpha(x)}\Phi^*(x), \qquad A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \ .$$

b) Recall that to have a direct particle interpretation of a scalar field $\Phi(x)$, we need $\langle \Omega|\Phi(0)|\Omega\rangle := v = 0$. At leading order, v is the value of the field that minimizes the corresponding scalar potential. In the theory given above, determine v from the potential for $m^2 > 0$ and $\lambda > 0$. When this theory is quantized with $Qe \sim \lambda \ll 1$, what kinds of particles does it describe? And what are their masses and how many real degrees of freedom do each of them have?

c) Suppose instead that $m^2 = -\mu^2 < 0$? What are the possible values of v?

d) When $m^2 < 0$, we need to shift $\Phi(x)$ by v to get a particle interpretation. To do this, it is convenient to apply a specific gauge transformation to make $\Phi(x)$ real everywhere, such that

$$\Phi(x) = v + h(x)/\sqrt{2}\ ,$$

where $h(x)$ is a real scalar and v is real as well (and a minimum of $V(\Phi, \Phi^*)$). Plug this form back into the original Lagrangian and simplify. What kinds of particles does this theory describe when it is quantized? And what are their masses and how many real degrees of freedom do each of them have?

e) Calculate the total unpolarized decay rate for $h \to AA$ in this theory assuming this decay is kinematically allowed.

f) Let $\Psi(x) = (\psi_L,\, \psi_R)^t$ be a Dirac fermion with $\psi_L(x) \to e^{iQ\,\alpha(x)}\psi_L(x)$ and $\psi_R(x) \to \psi_R(x)$ under gauge transformations. Show that a regular mass term $M\bar{\Psi}\Psi$ is not gauge invariant, but that instead the coupling

$$\mathscr{L} \supset -y(\Phi\bar{\psi}_R\psi_L + \Phi^*\bar{\psi}_L\psi_R)$$

is allowed and leads to a Ψ mass for $m^2 < 0$.

g) Compute the decay rate $h \to \Psi\bar{\Psi}$ assuming it is kinematically allowed.

5. QED Fun (**10 = 3+4+3 points**)

a) Find the amplitude for electron-electron elastic scattering at leading order in the electromagnetic coupling in QED.

b) Compute the summed and squared matrix element you would use to find the total unpolarized cross section for this process. Simplify until you are only left with traces over Dirac matrices.

c) Find the summed and squared matrix element you would use to calculate the total unpolarized cross section for Bhabha scattering. You may leave your result in terms of traces over Dirac matrices.