# PHYS 526 Notes #9: Quantum Electrodynamics

David Morrissey
November 18, 2012

Quantum Electrodynamics (QED), finally.

## 1 QED Lite

To get warmed up, let us first do some real calculations in the scalar fermion theory we studied previously,

$$\mathscr{L} = \bar{\Psi} i\gamma^\mu \cdot \partial_\mu \Psi - m\bar{\Psi}\Psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - y\phi\bar{\Psi}\Psi \ . \tag{1}$$

We will see that this theory is analagous to QED, with $\Psi$ representing the electron and the scalar playing a similar role to the photon.

***e.g. 1.*** **Decay of the Scalar**

The easiest physical process to calculate in this theory is the decay of the scalar to a fermion-antifermion pair, which is allowed by the interaction of Eq. (1) provided $M > 2m$. The relevant Feynman diagram (with time going from left to right) is shown in Fig. 1. Applying the Feynman rules, we get the amplitude for $p_0 \to (p_1, s_1) + (p_2, s_2)$

$$-i\mathcal{M} = -iy\, \bar{u}^c(p_2, s_2) v_c(p_1, s_1) \ . \tag{2}$$

The quantity we are usually the most interested in is the total decay width, where we sum over the partial decay widths to all possible final momenta and spins. Summing over spins is also the appropriate thing to do if our detector is unable to measure the spin of the outgoing fermions. Using our formula for the decay rate, we find

$$\Gamma = \frac{1}{2M}\int \frac{d^3p_1}{2E_1(2\pi)^3}\int \frac{d^3p_2}{2E_2(2\pi)^3}\,(2\pi)^4\delta^{(4)}(p_0 - p_1 - p_2)\ \text{``}|\mathcal{M}|^{2\prime\prime} \ , \tag{3}$$

where

$$\text{``}|\mathcal{M}|^{2\prime\prime} = \sum_{s_1=1}^{2}\sum_{s_2=1}^{2} |\mathcal{M}|^2 \ . \tag{4}$$

Having to sum over spins might seem tedious, but it will actually make our lives easier.[1] Our first task, however, is to take the complex conjugate of $\mathcal{M}$. We have

$$\mathcal{M}^*/y \;=\; [\bar{u}_2 v_1]^* = [u_2^\dagger \gamma^0 v_1]^\dagger = v_1^\dagger (\gamma^0)^\dagger u_2 = \bar{v}_1 u_2 \ . \tag{5}$$

[1] If we had wanted to find the result for specific spins, we would have had to use the explicit forms for $u$ and $\bar{u}$ and mutltiplied out the matrices.

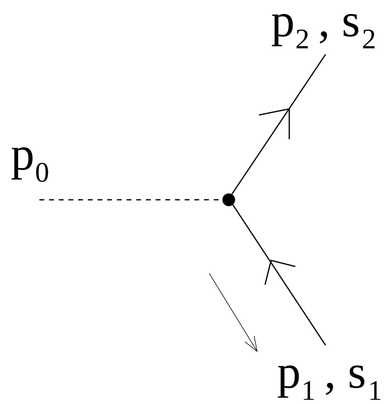


Figure 1: Feynman diagram for the decay of the scalar.

where we have used $(\gamma^0)^\dagger = \gamma^0$ and we have written this in matrix notation. Putting this into the spin sum and writing the Dirac indices explicitly,

$$\frac{1}{y^2}\sum_{s_1,s_2}|\mathcal{M}|^2 = \sum_{s_1,s_2}\bar{v}_1^a u_{2a}\bar{u}_2^b v_{1b} \tag{6}$$

$$= (\not{p}_2+m)_a{}^b \sum_{s_1} v_{1b}\bar{v}_1^a \tag{7}$$

$$= (\not{p}_2+m)_a{}^b(\not{p}_1-m)_b{}^a \tag{8}$$

$$= tr\left[(\not{p}_2+m)(\not{p}_1-m)\right] \tag{9}$$

$$= 4(p_1\cdot p_2)-4m^2\ . \tag{10}$$

In going to the second line we have used $\sum_s u(p,s)\bar{u}(p,s) = (\not{p}+m)$ and in going to the third line we have used $\sum_s v(p,s)\bar{v}(p,s) = (\not{p}-m)$. The Dirac index structure of the third line is precisely a trace, which is how we got the fourth line. And for the fifth line, we have used our Dirac trace tricks. This result is representative of what happens in general when we sum over spins - the Dirac structure reduces to one or more traces in the Dirac space. With this "summed and squared" amplitude "$|\mathcal{M}|^{2\prime\prime}$, it is just a matter of doing some integrals to get the total decay rate $\Gamma$.

***e.g. 2.*** **Fermion-Fermion Scattering**

As a second example, consider the elastic scattering of a pair of fermions mediated by the scalar. At leading non-trivial order, there are two distinct contributions to this process, shown in Fig. 2. Each diagram gives its own contribution to the amplitude, and we must sum them all up. The result is

$$-i\mathcal{M} = -i\mathcal{M}_t - i\mathcal{M}_u \tag{11}$$

$$= (-iy)^2\left[\frac{i}{(p_1-p_3)^2-M^2}\,\bar{u}_3^a u_{a1}\,\bar{u}_4^b u_{b2} - \frac{i}{(p_1-p_4)^2-M^2}\,\bar{u}_4^a u_{a1}\,\bar{u}_3^b u_{b2}\right]\ ,$$

where $u_{a1}$ stands for $u_a(p_1, s_1)$ and so on. The relative minus sign comes from the contraction structure, much like in the related example discussed in `notes-8`. Note that the two diagrams differ by the exchange of the fermions 3 and 4.

When we square the amplitude to get $|\mathcal{M}|^2$ there will be interference between the two contributions leading to four distinct terms. This can get cumbersome, but we can use the same methods as before to evaluate them. Putting this result into our formula for scattering cross sections, we can find the total (or differential) scattering cross section for a given values of the initial spins $s_1$ and $s_2$ the final spins $s_3$ and $s_4$. However, most scattering experiments are unpolarized, in that they do not have fixed values for the spins of the initial states. If this is the case, we should *average* the cross section over the spins of the initial states. Furthermore, if the experiment does not measure the spins of the final states, we should also sum the cross section over final spins. The expression for the *total unpolarized cross section* in this case is therefore given by the same expression we had previously for the total cross section, but with

$$|\mathcal{M}|^2 \to \text{``}|\mathcal{M}|^{2\prime\prime} = \frac{1}{4}\sum_{s_1,s_2}\sum_{s_3,s_4}|\mathcal{M}|^2\ , \tag{12}$$

where the factor of 1/4 comes from averaging over the $4 = 2\times 2$ possible initial spins.

To illustrate this, let's compute a few of the terms that arise. We have

$$|\mathcal{M}|^2 = |\mathcal{M}_t|^2 + \mathcal{M}_t\mathcal{M}_u^* + \mathcal{M}_t^*\mathcal{M}_u + |\mathcal{M}_u|^2\ . \tag{13}$$

The first term gives

$$\begin{aligned}
(t-m^2)^2y^2\times\sum_{s,s'}|\mathcal{M}_t|^2 &= \sum_{\{s\}}(\bar{u}_3^a u_{1a}\bar{u}_1^b u_{3b})(\bar{u}_4^c u_{2c}\bar{u}_2^d u_{4d}) && (14)\\
&= tr\left[(\not{p}_3+m)(\not{p}_1+m)\right]\ tr\left[(\not{p}_4+m)(\not{p}_2+m)\right] && (15)\\
&= 16\left[(p_1\cdot p_3)+m^2\right]\ \left[(p_2\cdot p_4)+m^2\right]\ . && (16)
\end{aligned}$$

The second term is a bit trickier. Here, it really helps to write out the Dirac indices:

$$\begin{aligned}
(t-m^2)(u-m^2)y^2\times\sum_{\{s\}}\mathcal{M}_t\mathcal{M}_u^* &= -\sum_{\{s\}}(\bar{u}_3^a u_{1a}\,\bar{u}_4^b u_{2b})(\bar{u}_1^c u_{4c}\,\bar{u}_2^d u_{3d}) && (17)\\
&= -(\not{p}_1+m)_a{}^c(\not{p}_4+m)_c{}^b(\not{p}_2+m)_b{}^d(\not{p}_3+m)_d{}^a && (18)\\
&= -tr\left[(\not{p}_1+m)(\not{p}_4+m)(\not{p}_2+m)(\not{p}_3+m)\right] && (19)
\end{aligned}$$

It is straightforward to compute this trace using our trace tricks and I will leave it to you. You should now also be able to compute the other two terms that contribute to the summed and squared amplitude.

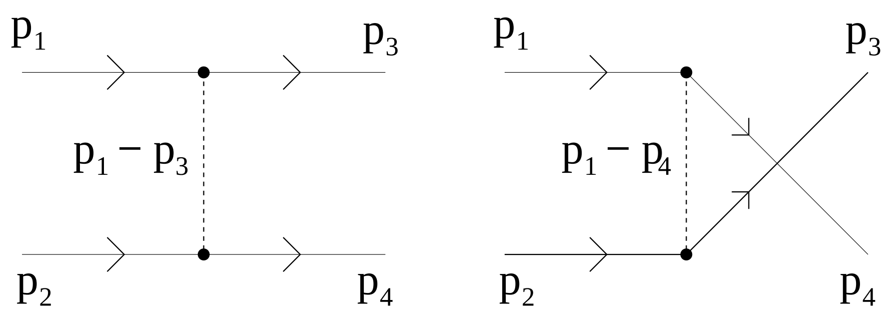


Figure 2: Feynman diagram for the elastic scattering of fermions.

# 2 QED

After this warmup, let us now present the Lagrangian for QED. It is

$$\mathscr{L} = \bar{\Psi} i\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{20}$$

where

$$D_\mu = \partial_\mu + ieQA_\mu \ , \tag{21}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{22}$$

That's it! We identify the fermion $\Psi$ with the electron and the vector field $A_\mu$ with the photon.[2] Indeed, the vector term coincides with the Lagrangian we had for classical electromagnetism in the absence of sources.

Our convention is to set $Q = -1$ and to take the coupling $e$ to be the magnitude of the electron charge. In our natural units it is dimensionless. The experimentally measured value is

$$\alpha := \frac{e^2}{4\pi} \simeq 1/(137.036) \ . \tag{23}$$

We will see that our pertubative expansion in the coupling corresponds to an expansion in $\alpha$. Since $\alpha \ll 1$, this expansion should converge very quickly.

## 2.1 Gauge Invariance

The QED Lagrangian has a global $U(1)$ symmetry under the transformations

$$\begin{cases} \Psi & \to & e^{iQ\alpha}\Psi \\ A_\mu & \to & A_\mu \end{cases} , \tag{24}$$

[2] We will add other fermions later on.

for any constant $\alpha$. The corresponding Noether current is

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \ . \tag{25}$$

On the other hand, suppose we're feeling adventurous and decide to elevate the transformation parameter to a function on spacetime: $\alpha = \alpha(x)$. Doing so, we find that the transformation above is no longer a symmetry of the Lagrangian. In particular,

$$\bar{\Psi}i\gamma^\mu\partial_\mu\Psi \to \bar{\Psi}i\gamma^\mu\partial_\mu\Psi + \bar{\Psi}i\gamma^\mu(iQ\partial_\mu\alpha)\Psi. \tag{26}$$

Evidently the transformation of Eq. (24) is *not* a symmetry of the theory for non-constant parameters $\alpha(x)$.

We can restore the invariance of the fermion terms if we also have the vector field transform according to:

$$\left\{ \begin{array}{ccc} \Psi & \to & e^{iQ\alpha}\Psi \\ A_\mu & \to & A_\mu - \frac{1}{e}\partial_\mu\alpha. \end{array} \right. \tag{27}$$

Together, this implies that

$$(\partial_\mu + ieQA_\mu)\Psi \equiv D_\mu\Psi \to e^{iQ\alpha}D_\mu\Psi, \tag{28}$$

and therefore $\Psi i\gamma^\mu D_\mu\Psi$ is invariant under the transformation for arbitrary $\alpha(x)$ provided the vector field also transforms as indicated. The differential operator $D_\mu$ is sometimes called a *covariant derivative*. Even better, if we look at the effect of this shift on the photon kinetic term, we find that it remains unchanged as well:

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \to (\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{e}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\alpha = F_{\mu\nu} + 0. \tag{29}$$

Thus, QED is invariant under the transformations of Eq. (27) for any reasonable arbitrary function $\alpha(x)$.

At first glance this invariance might just seem like a clever trick, but the river beneath these still waters runs deep. Thinking back to regular electromagnetism (of which QED is just the quantized version), one often deals with scalar and vector potentials. These potentials are not unique and are therefore not observable (for the most part), and the true "physical" quantities are the electric and magnetic fields. The invariance of the Lagrangian reflects this fact. These transformations are called *gauge transformations*, and the photon derived from $A_\mu$ is sometimes said to be a gauge boson.

Keeping in mind the story from electromagnetism, the interpretation of the quantum fields in QED is that ***only those quantities that are invariant under the transformations of Eq.*** (27) ***are physically observable.*** In particular, the vector field $A_\mu$ that represents the photon is not itself an observable quantity, but the gauge-invariant field strength $F_{\mu\nu}$ is. Put another way, the field variables we are using are redundant, and the transformations of Eq.(27) represent an *equivalence relation*: any two set of fields $(\Psi, A_\mu)$ related by such a transformation represent the same physical configuration. Sometimes the invariance under Eq. (27) is called a local or gauge symmetry, but it is not really a symmetry at all. A true symmetry implies that different physical configurations have the same properties. Gauge invariance is instead a statement about which configurations are physically observable.

Gauge invariance is also sensible if we consider the independent polarization states of the photon, of which there are two. The vector field $A_\mu$ represents the photon, but it clearly has four independent components. Of these, the timelike polarization component is already non-dynamical on account of the form of the vector kinetic term. Invariance under gauge transformations effectively removes the additional longitudinal polarization leaving behind only the two physical transverse polarization states. Note as well that if the photon had a mass term, $\mathscr{L} \supset m^2 A_\mu A^\mu/2$, the theory would no longer be gauge invariant. Instead, the longitudinal polarization mode would enter as physical degree of freedom. Equivalently, gauge invariance forces the photon to be massless.

In the discussion above we started with the QED Lagrangian and showed that it was gauge-invariant. However, the modern view is to take gauge invariance as the fundamental principle. Indeed, the only way we know of to write a consistent, renormalizable theory of interacting vector fields is to have an underlying gauge symmetry. For QED, we could have started with a local $U(1)$ gauge invariance for a charged fermion field and built up the rest of the Lagrangian based on this requirement. In this context, the vector field is needed to allow us to define a sensible derivative operator on the fermion field, which involves taking a difference of two fields at different spacetime points with apparently different transformation properties, and corresponds to something called a *connection*.[3] Gauge invariance completely fixes the photon-fermion interactions up to the overall charge $Q$, illustrating why it is so powerful.

## 2.2 Feynman Rules

It turns out that quantizing the vector field $A_\mu$ using the tools we've developed is a bit tricky on account of the gauge redundancy. We will come back to this a bit later. For now, we will simply present the result of the quantization procedure in the form of Feynman rules for the photon and the electron. This is all that is needed to compute scattering processes.

The photon field $A_\mu$ has a mode expansion similar to that of the fermion in that every momentum mode has a polarization associated to it. In this case, we need two (possibly complex) polarization basis vectors $\epsilon^\mu(p,\lambda)$, $\lambda = 1, 2$. These states are orthogonal,

$$\epsilon^\mu(p,\lambda)\epsilon^*_\mu(p,\lambda') = \delta^{\lambda\lambda'} \ , \tag{30}$$

and they satisfy a truncated completeness relation,

$$\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{\nu\,*}(p,\lambda) = -\eta^{\mu\nu} + (p^\mu \text{ stuff}) \ . \tag{31}$$

[3] See Ch.15 of Peskin & Schroeder for a nice explanation of these slightly cryptic comments.

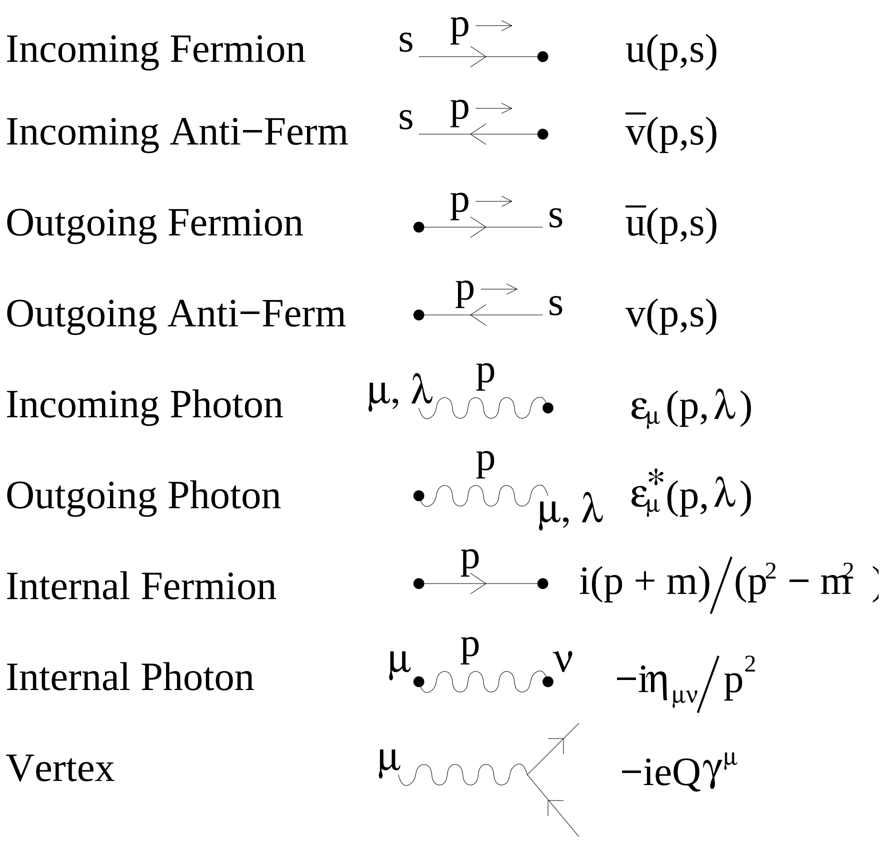


Figure 3: Feynman rules for QED.

We will see shortly that the $p^\mu$ stuff can be ignored. In terms of these polarization basis vectors, the mode expansion is

$$A^\mu(x) = \int \widetilde{dk} \, \left[a(k,\lambda)\epsilon^\mu(k,\lambda)e^{-ik\cdot x} + a^\dagger(k,\lambda)\epsilon^{\mu *}(k,\lambda)e^{ik\cdot x}\right] \ , \tag{32}$$

where $k^0 = \sqrt{\vec{k}^2}$.

Much like for external fermion states, an outgoing external photon will contribute a factor of $\epsilon^*_\mu(p,\lambda)$ to the amplitude, and an incoming photon will give $\epsilon_\mu(p,\lambda)$. Also like for fermions, the photon propagator takes the form

$$\frac{\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{\nu *}(p,\lambda)}{p^2 + i\epsilon} = \frac{i}{p^2 + i\epsilon}\left(-\eta^{\mu\nu} + p^\mu \text{ stuff}\right) \ . \tag{33}$$

Again, we will see that the $p^\mu$ stuff can be ignored.

Having made these introductions, we can now state the Feynman rules for QED. They are identical to those we had for QED Lite, but with photon stuff instead of scalar stuff. This leads to three essential differences that we summarize in Fig. 3. The first is that the photon-fermion-antifermion vertex is

$$\text{Vertex} = -iQe\gamma^\mu \ . \tag{34}$$

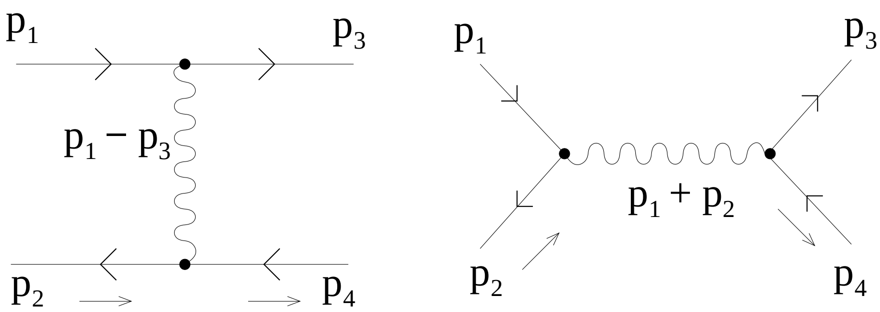


Figure 4: Diagrams for Bhabha scattering at leading order.

The second is that we use a photon propagator (dropping the $p^\mu$ stuff) instead of the scalar propagator. And third, external photon lines contribute polarization vectors $\epsilon^{(*)}(p,\lambda)$ to the value of the diagram. At the end of the day, all vector indices should be contracted.

# 3 Computing Stuff

Knowing how to compute things in QED is the first step to doing research in particle physics. These skills also transfer over to many other areas of physics. We present here a few specific examples of standard QED processes.

***e.g. 3.*** **Electron-Positron (Bhabha) Scattering**

The antiparticle of the electron, the antielectron, is sometimes also called the positron. At leading order, there are two diagrams for this process shown in Fig. 4. The amplitude is therefore

$$\begin{aligned} -i\mathcal{M} &= -i(\mathcal{M}_t + \mathcal{M}_s) \qquad (35) \\ &= (-ieQ)^2 \left[ (\bar{u}_3\gamma^\mu u_1)\,(\bar{v}_2\gamma^\nu v_4) \frac{-i\eta_{\mu\nu}}{(p_1-p_3)^2 + i\epsilon} + (\bar{v}_2\gamma^\mu u_1)\,(\bar{u}_3\gamma^\nu v_4)\,\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2+i\epsilon}\right] . \end{aligned}$$

The relative sign between the two pieces requires a slightly careful examination of the contraction structure. It is straightforward to square this and sum over spins.

***e.g. 4.*** **Compton Scattering**

Compton scattering corresponds to electron-photon scattering, $e\gamma \to e\gamma$. There are two diagrams at leading order, shown in Fig. 5. They are equal to

$$\begin{aligned} -i\mathcal{M} &= -i(\mathcal{M}_s + \mathcal{M}_t) \qquad (36) \\ &= (-ieQ)^2 \left[ \bar{u}_3\gamma^\mu \frac{i(\not{q}+m)}{q^2-m^2}\gamma^\nu u_1 + \bar{u}_3\gamma^\nu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\mu u_1 \right] \epsilon^*_\mu(p_4)\,\epsilon_\nu(p_2) \ , \end{aligned}$$

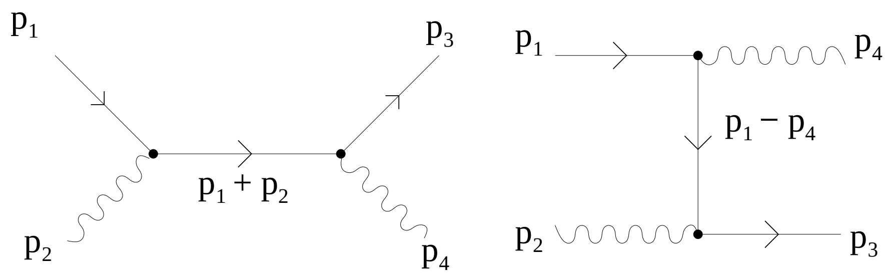


Figure 5: Diagrams for Compton scattering at leading order.

where $q = (p_1 + p_2)$ and $k = (p_1 - p_4)$. Note the specific contraction of indices and that we made sure to use the same momentum label for the distinguishable outgoing particles.

To illustrate how to deal with external photons, let us look at the summed and squared amplitude. In many experiments, we only measure the total rate and not the specific polarizations of the outgoing photons (or the fermion spins). Furthermore, the initial beams often have no net polarization. In this case, we should average over initial polarizations and spins and sum over final ones. The summed and squared matrix element that goes into the formula for the total unpolarized cross section is therefore

$$\text{“}|\mathcal{M}|^{2\prime\prime} = \frac{1}{2}\frac{1}{2}\sum_{s_1,s_3}\sum_{\lambda_2\lambda_4}|\mathcal{M}(s_1,s_3,\lambda_2,\lambda_4)|^2\ , \tag{37}$$

where we have divided by the number of intial spins (2) and initial polarizations (2). The $s$-channel contribution is

$$\text{“}|\mathcal{M}_s|^{2\prime\prime} \propto \sum_{s_1,s_3}[\bar{u}_3\gamma^\mu(\not{q}+m)\gamma^\nu u_1]\,[\bar{u}_1\gamma^\beta(\not{q}+m)\gamma^\alpha u_3]\sum_{\lambda_2,\lambda_4}(\epsilon^*_{\mu 4}\epsilon_{\alpha 4})\,(\epsilon_{\nu 2}\epsilon^*_{\beta 2}) \tag{38}$$

$$= tr[(\not{p}_3+m)\gamma^\mu(\not{q}+m)\gamma^\nu(\not{p}_1+m)\gamma^\beta(\not{q}+m)\gamma^\alpha]\,(-\eta_{\mu\alpha})(-\eta_{\nu\beta})\ . \tag{39}$$

This is pretty complicated, but it can be simplified with lots of work or by using clever tricks. A useful one is $\not{p}\,u(p) = m\,u(p)$. The $t$-channel piece goes through similarly.

# References


[1] M. E. Peskin and D. V. Schroeder, “An Introduction To Quantum Field Theory,” *Reading, USA: Addison-Wesley (1995) 842 p*

[2] M. Srednicki, “Quantum field theory,” *Cambridge, UK: Univ. Pr. (2007) 641 p*