

# PHYS 526 Notes #1: Classical Fields

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A field is just a continuous system defined in time and space. You have already encountered a number of fields in your physics career.

## *e.g.* 1. Some Fields

$T(t, \vec{x}) := T(x)$  = temperature at every point in the room.

$\phi(x)$  = electric potential at  $x = (t, \vec{x})$

$\vec{E}(x)$  = electric field at  $x = (t, \vec{x})$

In this example, the first and second cases are scalar fields, where the field specifies a single (real) number at every point in spacetime. The third case, the electric field, is a three-vector field specifying a vector at every point.

To describe a system characterized by one or more fields, we must specify the equations of motion and the boundary conditions. A convenient way to encode the equations of motion is in terms of a Lagrangian. In the next example, we will illustrate how to model a continuous system as the limit of a discrete one. This example should also give you some intuition for where the various terms that usually appear in field theory Lagrangians come from. After this, we will look at how to derive the equations of motion directly, then specialize to relativistic systems, and then move on to Hamiltonians and symmetries.

## 1 Discretized Example: A String with Free Ends

Consider a string of length  $\ell$  whose ends can move freely. The configuration of the string is described by  $h(t, x)$ , the height of the string element at point  $x$  and time  $t$ . Since the string has free ends, we also have the boundary conditions  $\partial_x h(t, 0) = 0 = \partial_x h(t, \ell)$ . We will model the system by  $N$  equally-spaced masses  $m$  connected by springs  $k$ , with each mass only allowed to move vertically. The setup is illustrated in Fig. 1. We will write

$$\Delta = \ell/N, \quad x_n = n\Delta, \tag{1}$$

and describe the height of the  $n$ -th mass by

$$h_n(t) = h(t, x_n). \tag{2}$$

These are the  $N$  generalized coordinates of the discrete system.

The total kinetic energy is

$$T = \sum_{n=1}^N \frac{1}{2} m \dot{h}_n^2. \tag{3}$$

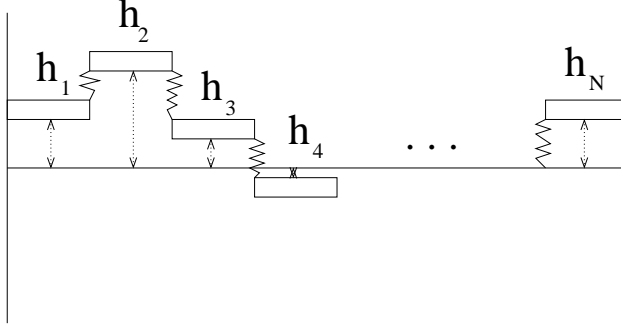


Figure 1: Discretized string.

For the potential energy, it will be the sum of the spring displacements between each of the mass elements. Thus, we have

$$V = \sum_{n=1}^{N-1} \frac{1}{2} k (h_{n+1} - h_n)^2 . \quad (4)$$

Let us now try to connect with the continuous case by taking  $N \rightarrow \infty$  (and  $\Delta \rightarrow 0$ ) while keeping  $\rho := m/\Delta$  and  $\kappa := k\Delta$  fixed. The kinetic energy becomes

$$T = \sum_n \Delta \frac{1}{2} \frac{m}{\Delta} \dot{h}_n^2 \quad (5)$$

$$\rightarrow \int_0^L dx \frac{1}{2} \rho \left( \frac{\partial h}{\partial t} \right)^2 , \quad (6)$$

where  $h = h(t, x)$ , and we have used the formal definition of an integral. For the potential energy, we get

$$V = \sum_n \Delta \frac{1}{2} (k\Delta) \frac{(h_{n+1} - h_n)^2}{\Delta^2} \quad (7)$$

$$\rightarrow \int_0^L dx \frac{1}{2} \kappa \left( \frac{\partial h}{\partial x} \right)^2 . \quad (8)$$

We now have a Lagrangian  $L = T - V$  for the continuous system.

The discretized form of the theory can also be used to deduce the equations of motion for the continuous version. Remembering that the degrees of freedom are  $h_n(t) = h(t, x_n)$ , we find

$$\partial_t \left( \frac{\partial L}{\partial \dot{h}_n} \right) - \frac{\partial L}{\partial h_n} = \partial_t (\Delta \rho \dot{h}_n) + \frac{\kappa}{\Delta} [(h_n - h_{n-1}) - (h_{n+1} - h_n)] \quad (9)$$

$$\rightarrow \Delta \rho \left( \partial_t^2 - \frac{\kappa}{\rho} \partial_x^2 \right) h(t, x) \quad (10)$$

Thus, we find the equation of motion for  $h(t, x)$  is

$$\left( \partial_t^2 - \frac{\kappa}{\rho} \partial_x^2 \right) h(t, x) = 0 . \quad (11)$$

This is a familiar wave equation. A simple solution is

$$h(t, x) = a(k) e^{-i\omega t} e^{ipx} + (c.c.) , \quad (12)$$

with  $\omega$  related to  $p$  by the *dispersion relation*

$$\omega = \pm p \left( \frac{\kappa}{\rho} \right)^{1/2} . \quad (13)$$

These solutions correspond to left- and right-moving waves with wavenumber  $k$  and equal phase and group velocities  $v_p = \sqrt{\kappa/\rho} = v_g$ . Applying the boundary conditions restricts the possible values of  $p$ .

A simple generalization of this example is to add the term  $M^2 h^2/2$  to  $V$ . Physically, this would represent attaching each mass element to the  $h = 0$  line by a spring with constant  $M^2$ . (Note that  $M^2$  has nothing at all to do with the mass  $m$  of the elements.) The equation of motion with this extra term becomes

$$\left( \partial_t^2 - \frac{\kappa}{\rho} \partial_x^2 + M^2 \right) h(t, x) = 0 . \quad (14)$$

The solution of Eq. (12) still works here, but with a different dispersion relation:

$$\omega = \pm \left( \frac{\kappa}{\rho} \right)^{1/2} \sqrt{p^2 + (\rho/\kappa) M^2} . \quad (15)$$

This should remind you of the energy of a single relativistic particle.

## 2 Equations of Motion (Lagrangian)

In the example above, we started with a continuous system, modelled it with a discrete one, and took a limit to get back to the continuous case. This works, but it is also very tedious. Instead, it is useful to apply the calculus of variations to continuous field systems directly. Suppose our system is described by the field  $\phi(t, x)$  defined in one spatial dimension. In the Lagrangian formulation, we should think of the  $x$  parameter as a continuous label of generalized coordinates:

$$\phi(t, x) = q_x(t) . \quad (16)$$

Consider an action for the system of the form

$$S[\phi] = \int_{t_1}^{t_2} dt \left[ \int_{x_1}^{x_2} dx \mathcal{L}(\phi, \partial_t \phi, \partial_x \phi) \right] . \quad (17)$$

Note that  $L = \int dx \mathcal{L}$  here. We call  $\mathcal{L}$  the *Lagrangian density*, and we will see that the Lagrangian must take this form if we are to maintain causality.

To derive the equations of motion, we apply the principle of least action, namely that  $\delta S = 0$  under any variation  $\phi(t, x) \rightarrow \phi(t, x) + \delta\phi(t, x)$  with  $\delta\phi(t_1, x) = \delta\phi(t_2, x) = 0$ . This implies

$$\delta S = \int dt \int dx \left[ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \delta(\partial_x \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta(\partial_t \phi) \right] \quad (18)$$

$$\begin{aligned} &= \int dt \int dx \delta\phi \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_t \left[ \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right] - \partial_x \left[ \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right] \right) \\ &\quad - \int dt \int dx \partial_x \left[ \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \delta\phi \right] . \end{aligned} \quad (19)$$

In the second line, we have integrated by parts over both  $t$  and  $x$ . The boundary term in the  $t$  part vanishes due to our assumption that  $\delta\phi(t_1, x) = 0 = \delta\phi(t_2, x)$ , but we are still left with the  $x$  part which produces the last term in Eq. (19). For all the systems to be considered in this course, we will assume spatial boundary conditions such that this term vanishes ( $\phi(t, x_{1,2}) = 0$  or  $\partial_x \phi(t, x_{1,2}) = 0$ ), but you should be aware of its existence.

With that slight complication out of the way, and remembering that  $\delta\phi$  is arbitrary, we find the equations of motion to be:

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_t \left[ \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right] - \partial_x \left[ \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right] . \quad (20)$$

These apply for any value of  $t$  and  $x$  in the allowed range. It should also be clear that plugging the Lagrangian density from the example into these equations of motion reproduces Eq. (11) or Eq. (14), depending on the choice of  $V$ .

The derivation given here also generalizes to systems with multiple fields defined in any number of spacetime dimensions. The Lagrangian for  $n$  spacetime dimensions ( $(n-1)$  spatial dimensions) becomes

$$L = \int d^{(n-1)}x \mathcal{L}(\phi_i, \partial_\mu \phi_i) , \quad (21)$$

and the equations of motion are

$$0 = \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right] . \quad (22)$$

Note that we have used four-vector indices (generalized to  $n$  spacetime dimensions) to simplify the notation.

The variational principle applied to fields gives rise to the idea of functional derivatives. In  $n$  spacetime dimensions (one time,  $(n-1)$  space) we define

$$\frac{\delta\phi(x_a)}{\delta\phi(x_b)} = \delta^{(n)}(x_a - x_b) = \delta(t_a - t_b) \delta^{(n-1)}(\vec{x}_a - \vec{x}_b) . \quad (23)$$

Note that we write  $x_a = (t_a, \vec{x}_a)$  and so on. This functional derivative is just a continuous generalization of  $\partial_\mu x^\nu = \delta_\mu^\nu$ . We also demand that the chain rule (and product rule) hold:

$$\frac{\delta f(\phi(x))}{\delta \phi(x')} = \frac{\partial f}{\partial \phi} \frac{\delta \phi(x)}{\delta \phi(x')} . \quad (24)$$

The functional derivative should also commute with spacetime derivatives:

$$\frac{\delta}{\delta \phi(x')} [\partial_\mu \phi(x)] = \partial_\mu \frac{\delta \phi(x)}{\delta \phi(x')} . \quad (25)$$

If you don't like this, you can usually avoid it by integrating by parts.

In terms of functional derivatives (and assuming that boundary terms vanish), the equations of motion can be written as

$$0 = \frac{\delta S[\phi]}{\delta \phi(x)} . \quad (26)$$

This is easier to remember than varying  $\phi$  by hand.

### *e.g.* 2. Action in $n$ spacetime dimensions

Consider the action

$$\begin{aligned} S[\phi] &= \int dt \int d^{(n-1)}x \left[ \frac{1}{2}(\partial_t \phi)^2 - \frac{\xi}{2}(\vec{\nabla} \phi)^2 - f(\phi) \right] \\ &= \int dt \int d^{(n-1)}x \left[ \frac{1}{2} \phi \left( -\partial_t^2 + \xi \vec{\nabla}^2 \right) \phi - f(\phi) \right] , \end{aligned}$$

where we have integrated by parts in the second line. Taking a functional derivative gives

$$\begin{aligned} \frac{\delta S[\phi]}{\delta \phi(x')} &= \int dt \int d^{(n-1)}x \left[ \left( -\partial_t^2 + \xi \vec{\nabla}^2 \right) \phi - \frac{\partial f(\phi)}{\partial \phi} \right] \delta^{(n)}(x - x') \\ &= \left( -\partial_t^2 + \xi \vec{\nabla}^2 \right) \phi(x') - \frac{\partial f(\phi)}{\partial \phi}(x') . \end{aligned}$$

This is precisely what we would have obtained from the principle of least action. Note that  $x$  in the integration is a dummy variable (in that we can rename it anything we want), while  $x'$  is fixed. The quantity in the last line is a function of  $x'$ .

## 3 Relativistic Actions

Much of this course will focus on systems that are invariant under special relativity (boosts and rotations) and that are defined on all spacetime. The basic theory we will consider for

the first part of this course is that of a single real scalar field defined on all of spacetime ( $t, x^i \in (-\infty, \infty)$ ) with action

$$S = \int d^4x \left[ \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi) \right] \quad (27)$$

$$= \int d^4x \left[ \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi) \right], \quad (28)$$

where  $d^4x = dt d^3x$ , and we will often abbreviate  $\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) = (\partial\phi)^2$ . We will also assume that  $\phi$  and  $\partial_\mu\phi$  both vanish at spacetime infinity.

Compared to the previous example, *e.g.* 2, we have  $\xi = 1$ . Only in this case can we write the time and space derivatives as  $(\partial\phi)^2$ . This is essential for the theory to be Lorentz invariant (as we will see later). We could have also included other fully-contracted terms such as  $\phi^{23}\partial^2\phi^5\partial_\mu\partial_\nu\phi\partial^\mu\phi\partial^\nu\phi$ . There is nothing wrong with them, but for now, let us start with the most simple possibility.

The action has been also written in terms of a Lagrangian density, with terms that only depend on the single local variable  $x$ . In principle, the action could also contain terms like

$$S \supset \int d^4x d^4y \phi(x)\phi(x-y). \quad (29)$$

However, this would imply that stuff at point  $x$  can instantaneously affect stuff at point  $y$  far away. This is not consistent with causality (or locality). Thus we will only consider local actions that can be written in terms of a Lagrangian density.

Let's find solutions to the equations of motion for

$$V(\phi) = \frac{1}{2}m^2\phi^2. \quad (30)$$

Taking a functional derivative of the action, we find

$$(-\partial^2 - m^2)\phi = 0. \quad (31)$$

This is called the Klein-Gordon equation. A solution is

$$\phi(x) = a(k)e^{-ik\cdot x} \quad (32)$$

with  $k^0 = \pm\sqrt{\vec{k}^2 + m^2}$ . This looks like the relativistic relation between energy and momentum. The most general solution is obtained by taking linear combinations of these specific solutions and making sure the result is real (since  $\phi(x)$  is a real-valued field):

$$\phi(x) = \int \widetilde{dk} \left[ a(\vec{k})e^{-ikx} + a^*(\vec{k})a^{ikx} \right], \quad (33)$$

with  $k^0 = +\sqrt{\vec{k}^2 + m^2}$  and

$$\widetilde{dk} = \frac{d^3k}{2k^0(2\pi)^3}. \quad (34)$$

This funny combination has been chosen to obtain an integration measure that is Lorentz invariant. While the basic  $d^3k$  measure is not, the following combination is:

$$\int d^4k \Theta(k^0) \delta(k^2 - m^2) = \int d^3k \frac{1}{2k^0}, \quad k^0 = \sqrt{\vec{k}^2 + m^2}. \quad (35)$$

Another relativistic action that we will study down the road is that for electromagnetism. We will write the electric and vector potentials as a single four-vector:

$$A^\mu = (\phi, \vec{A}). \quad (36)$$

Let us now define the field strength tensor to be

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (37)$$

Note that it is antisymmetric:  $F_{\mu\nu} = -F_{\nu\mu}$ . The action for electromagnetism is

$$S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right). \quad (38)$$

This is invariant under Lorentz transformations because all the indices are contracted. The equations of motion derived from this action turn out to be Maxwell's Equations (with no sources)!

A complication that arises in electromagnetism is that the basic variables we have used to describe the theory,  $A^\mu$ , are not unique. In particular, transforming  $A^\mu$  by

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha, \quad (39)$$

leads to the same electric and magnetic fields for any function  $\alpha(x)$ . Fortunately, such transformations do not change  $F_{\mu\nu}$ , and therefore we obtain the same equations of motion. The physical interpretation is that the  $A^\mu$  are redundant variables, with different values able to describe the same physical configuration. This implies that all physical observables should be unaffected by the *gauge transformation* of Eq. (39).

## 4 Equations of Motion (Hamiltonian)

To connect with quantum mechanics later on, it will be useful to know the Hamiltonian formulation of our simple field theory. A sensible generalization of conjugate momenta to continuous systems is

$$\Pi(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(t, \vec{x})}. \quad (40)$$

The Hamiltonian is then

$$H = \int d^3x \mathcal{H} = \int d^3x \Pi(t, \vec{x}) \dot{\phi}(t, \vec{x}) - \mathcal{L}. \quad (41)$$

Not surprisingly, we call  $\mathcal{H}$  the Hamiltonian density. We can also generalize Poisson brackets:

$$\{f, g\} = \int d^3x'' \left[ \frac{\delta' f}{\delta' \phi(t'', \vec{x}'')} \frac{\delta' g}{\delta' \Pi(t'', \vec{x}'')} - \frac{\delta' g}{\delta' \phi(t'', \vec{x}'')} \frac{\delta' f}{\delta' \Pi(t'', \vec{x}'')} \right] \quad (42)$$

where the prime on the functional derivatives means that we don't touch the time variable in the sense  $\delta' \phi(t, \vec{x}) / \delta' \phi(t, \vec{y}) = \delta^{(3)}(\vec{x} - \vec{y})$ .

The Hamiltonian equations of motion are then

$$\dot{\phi}(t, \vec{x}) = \{\phi(t, \vec{x}), H\}, \quad \dot{\Pi}(t, \vec{x}) = \{\Pi(t, \vec{x}), H\}. \quad (43)$$

We also find that

$$\{\phi(t, \vec{x}), \phi(t, \vec{x}')\} = 0 = \{\Pi(t, \vec{x}), \Pi(t, \vec{x}')\} \quad (44)$$

$$\{\phi(t, \vec{x}), \Pi(t, \vec{x}')\} = \delta^{(3)}(\vec{x} - \vec{x}'). \quad (45)$$

Note that these relations only hold when both arguments are evaluated at the same value of  $t$ . (This is also true in the discrete case.)

Let's now apply this formalism to our basic scalar field theory. The conjugate momenta are

$$\Pi(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(t, \vec{x})} = \dot{\phi}(t, \vec{x}), \quad (46)$$

and the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} \Pi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + V(\phi). \quad (47)$$

The equation of motion for  $\phi$  yields

$$\dot{\phi}(t, \vec{x}) = \{\phi(t, \vec{x}), H\} \quad (48)$$

$$= \int d^3x' \left[ \frac{\delta' \phi(t, \vec{x})}{\delta' \phi(t, \vec{x}')} \frac{\delta' H}{\delta' \Pi(t, \vec{x}')} - \frac{\delta' \phi(t, \vec{x})}{\delta' \Pi(t, \vec{x}')} \frac{\delta' H}{\delta' \phi(t, \vec{x}')} \right] \quad (49)$$

$$= \int d^3x' \delta^{(3)}(\vec{x} - \vec{x}') \Pi(t, \vec{x}') \quad (50)$$

$$= \Pi(t, \vec{x}). \quad (51)$$

From the equation of motion for  $\Pi$  we get

$$\dot{\Pi}(t, \vec{x}) = \{\Pi(t, \vec{x}), H\} \quad (52)$$

$$= \int d^3x' \left[ \frac{\delta' \Pi(t, \vec{x})}{\delta' \phi(t, \vec{x}')} \frac{\delta' H}{\delta' \Pi(t, \vec{x}')} - \frac{\delta' \Pi(t, \vec{x})}{\delta' \Pi(t, \vec{x}')} \frac{\delta' H}{\delta' \phi(t, \vec{x}')} \right] \quad (53)$$

$$= \int d^3x' \left[ 0 - \delta^{(3)}(\vec{x} - \vec{x}') \left( -\vec{\nabla}^2 \phi + \frac{\partial V}{\partial \phi} \right) \right] \quad (54)$$

$$= \vec{\nabla}^2 \phi - \frac{\partial V}{\partial \phi}. \quad (55)$$

This is equivalent to what we found in the Lagrangian formulation.



## 5 Symmetries and Noether's Theorem

Symmetries play a key role in physics, and they have interesting implications for field theories (both classical and quantum). A symmetry is a transformation of the system that leaves the physics the same. For classical theories, “the same physics” means that the equations of motion should have the same form both before and after the transformation. We will consider both discrete and continuous symmetries, but the continuous case will turn out to be more interesting.

### *e.g.* 3. A discrete symmetry of our simple scalar theory.

Recall that the Lagrangian (density) was  $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$ . This is clearly unaffected by  $\phi \rightarrow \phi' = -\phi$ , and the equations of motion for  $\phi'$  are identical to those for  $\phi$ . On the other hand, this would not be a symmetry if  $V$  contained a term proportional to  $\phi^3$ .

A necessary and sufficient condition for a transformation to be a symmetry is that the action written in terms of the transformed coordinates take the same form as the original theory. There are two ways to think about this. The first is the *active picture*, where the configuration of the system is changed. For an action that is a functional of a set of fields  $\{\phi_i\}$ , this means that we explicitly change the fields

$$\phi_i \rightarrow \phi'_i = f_i(\phi) . \quad (56)$$

In turn, this changes the action according to

$$S[\phi] \rightarrow S[\phi'] := S'[\phi] , \quad (57)$$

where  $S'[\phi]$  is some new functional of the original (untransformed) fields  $\phi_i$ . In this active picture, a transformation is a symmetry if and only if

$$S'[\phi] = S[\phi] . \quad (58)$$

A standard example of an active transformation is performing a rotation on the system.

The second way to think of symmetries is in terms of the *passive picture*. Here, the idea is that we don't actually modify the system, but rather that we just use different variables to describe it. Suppose  $\phi_i = f_i(\phi')$ , for some new set of fields  $\{\phi'_j\}$ . Rewriting the action in terms of these new field variables, we find

$$S[\phi] = S[f(\phi')] = S'[\phi'] . \quad (59)$$

Such a transformation is a symmetry of the system if and only if

$$S'[\phi'] = S[\phi'] . \quad (60)$$

The standard example of a passive transformation is choosing a rotated set of axes to describe the system. Relative to the new axes, the coordinates that describe the system are different even though the system has not changed at all. You should convince yourself that both of these pictures are equivalent.

**e.g. 4. A continuous symmetry for two fields.**

Consider a theory with two real fields  $\phi_1$  and  $\phi_2$ :

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} [(\partial\phi_1)^2 + (\partial\phi_2)^2] - \frac{1}{2}m^2 (\phi_1^2 + \phi_2^2) \\ &= \frac{1}{2}(\partial\phi)^t(\partial\phi) - \frac{1}{2}m^2\phi^t\phi ,\end{aligned}$$

where  $\phi = (\phi_1, \phi_2)^t$ . This is clearly invariant under the discrete transformation  $\phi \rightarrow -\phi$ . It is also unchanged under transformations of the form

$$\phi \rightarrow \phi' = \mathcal{O}\phi , \quad (61)$$

where  $\mathcal{O}$  is any  $2 \times 2$  orthogonal matrix – satisfying  $\mathcal{O}^t\mathcal{O} = \mathbb{I}$ . These are called continuous transformations because they can be parametrized by a continuous parameter. In particular, up to a few signs, we can write any orthogonal matrix  $\mathcal{O}$  in terms of the parameter  $\alpha$ :

$$\mathcal{O} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} . \quad (62)$$

Note as well that this is the active picture, and we have just argued that  $S[\phi] \rightarrow S[\mathcal{O}\phi] = S'[\phi] = S[\phi]$ . Alternatively, in the passive picture we would write

$$\phi = \mathcal{O}\phi'' , \quad (63)$$

for some new field coordinates  $\phi''$ . Now,  $S[\phi] = S[\mathcal{O}\phi'']$  by assumption, but the fact that this is a symmetry corresponds to  $S[\mathcal{O}\phi''] := S''[\phi''] = S[\phi'']$ .

Continuous symmetries are especially interesting because they imply conservation laws. This relationship is called *Noether's theorem*. Let's derive it using the active picture. Consider a continuous (active) transformation parametrized by the parameter  $\alpha$ :

$$\phi_i(x) \rightarrow \phi'_i(x) = \phi_i(x) + \alpha\Delta\phi_i(x) , \quad (64)$$

where we will treat  $|\alpha| \ll 1$  and work to linear order. This will change the action by changing the Lagrangian, which we assume can be written as a Lagrangian density. For this to be a symmetry we must have  $S[\phi'] := S'[\phi] = S[\phi]$ . Equivalently, the Lagrangian density should obey

$$\mathcal{L}(\phi') := \mathcal{L}'(\phi) = \mathcal{L}(\phi) + \alpha\partial_\mu K^\mu . \quad (65)$$

Note that the Lagrangian density can change by a total divergence; this will only produce a vanishing surface term in the action, and will therefore yield the same equations of motion.

Plugging the form of Eq. (64) into Eq. (65), we obtain

$$0 = \mathcal{L}(\phi') - \mathcal{L}(\phi) - \alpha \partial_\mu K^\mu \quad (66)$$

$$= \sum_i \frac{\partial \mathcal{L}}{\partial \phi_i} \alpha \Delta \phi_i + \sum_i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \partial_\mu (\alpha \Delta \phi_i) - \alpha \partial_\mu K^\mu \quad (67)$$

$$= \sum_i \left[ \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right] \alpha \Delta \phi_i + \alpha \partial_\mu \left[ \sum_i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \Delta \phi_i - K^\mu \right], \quad (68)$$

where we have used the product rule to move derivatives around. The terms in the first part of the last line above vanish by the equations of motion. This implies that

$$0 = \partial_\mu j^\mu, \quad \text{where} \quad j^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \Delta \phi_i - K^\mu. \quad (69)$$

We say that the four-vector  $j^\mu$  is conserved. The reason for this is that if we define the corresponding *conserved charge* (not necessarily electric charge!) by

$$Q = \int d^3x j^0, \quad (70)$$

we find that

$$\partial_t Q = \int d^3x \partial_t j^0 = \int d^3x \vec{\nabla} \cdot \vec{j} = 0. \quad (71)$$

Note that we get zero because everything vanishes on the boundary, by assumption. The physical interpretation of  $j^\mu = (j^0, \vec{j})$  is that  $j^0$  is a charge density and  $\vec{j}$  is a current density.

### *e.g.* 5. More of our two-field example

Let's apply this general result to the theory considered in *e.g.* 4. Here, we find that  $\mathcal{L}(\phi') = \mathcal{L}(\phi)$ , so that  $K^\mu = 0$ . For small rotation angles  $\alpha$ , we find

$$\phi \rightarrow \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \phi + \alpha \begin{pmatrix} -\phi_2 \\ \phi_1 \end{pmatrix}. \quad (72)$$

Thus,  $\Delta \phi_1 = -\phi_2$  and  $\Delta \phi_2 = \phi_1$ . The conserved current is therefore

$$j^\mu = -(\partial^\mu \phi_1) \phi_2 + \phi_1 (\partial^\mu \phi_2). \quad (73)$$

It is straightforward to check that this current is indeed conserved.

A particularly important set of continuous transformations are the spacetime translations,

$$x^\lambda \rightarrow x^\lambda + a^\lambda. \quad (74)$$

For just about all the theories we will study in this course, these translations will be symmetries of the system. For now, let's look specifically at our simple scalar theory. It is

easiest to think of the translations as an active shift of the system:  $\phi(x) \rightarrow \phi'(x) = \phi(x + a)$  (with the integrals and derivatives in the action unchanged, possible because we integrate over all spacetime). For infinitesimal  $a^\lambda$ , we have

$$\phi'(x) = \phi(x) + a^\lambda \partial_\lambda \phi . \quad (75)$$

Applying this to the Lagrangian, we find

$$\mathcal{L}(\phi') = \mathcal{L}(\phi) + a^\lambda \partial_\mu (\delta_\lambda^\mu \mathcal{L}) . \quad (76)$$

Thus, spacetime translations are a symmetry of our theory with  $K_\lambda^\mu = \delta_\lambda^\mu \mathcal{L}$ . Applying our general result, the corresponding conserved currents are

$$j_{\mu\lambda} = \partial_\mu \phi \partial_\lambda \phi - \eta_{\mu\lambda} \mathcal{L} . \quad (77)$$

At this point, let me emphasize that we just considered four different symmetries at once; each value of  $\lambda = 0, 1, 2, 3$  corresponds to a different transformation.<sup>1</sup> In contrast,  $\mu$  labels the spacetime index that we always get on the current. However, since  $j_{\mu\lambda} = j_{\lambda\mu}$  in this case we can afford to be careless with the indices.

If we focus on spacetime translations, we should set  $\lambda = 0$ . The corresponding charge density is

$$j_{00} = \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\vec{\nabla} \phi)^2 + V(\phi) . \quad (78)$$

This is just the Hamiltonian density  $\mathcal{H}$  we derived previously (with  $\Pi = \partial_t \phi$ ). Thus, invariance under time translations corresponds to energy conservation. Similarly, for spacial translations the related charge density is

$$j^{0i} = -\partial_t \phi \partial_i \phi , \quad (79)$$

corresponding to a conserved spatial momentum density. Note, however, that the conserved spatial momentum density does not coincide with the conjugate momentum  $\Pi = \partial_t \phi$ . In general, these two quantities are not related to each other. Given the physical interpretation of  $j_{\mu\lambda}$ , it is given a special symbol

$$T^{\mu\nu} = j^{\mu\nu} , \quad (80)$$

and is usually called the *energy-momentum tensor*.

## References

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<sup>1</sup>This is why I used  $\lambda$  instead of  $\nu$ .